$\boxed{9.10}$ $$f_{X}(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{4}{5}\cdot(x+xy+y)\,dy =$$

$$= \frac{4}{5}\cdot\left[xy + x\cdot y^2/2 + y^2/2\right]_{y=0}^{y=1} = \frac{4}{5}\cdot\left(x+\frac{1}{2}x+\frac{1}{2}\right)\cdot \mathbb{1}[0<x<1]$$

SYMMETRY: $$f_{Y}(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{4}{5}\cdot\left(y+\frac{1}{2}y+\frac{1}{2}\right)\cdot\mathbb{1}[0<y<1]$$

$f_{X}(x)\cdot f_{Y}(y) \not\equiv f(x,y)$, THUS X AND Y ARE NOT INDEPENDENT

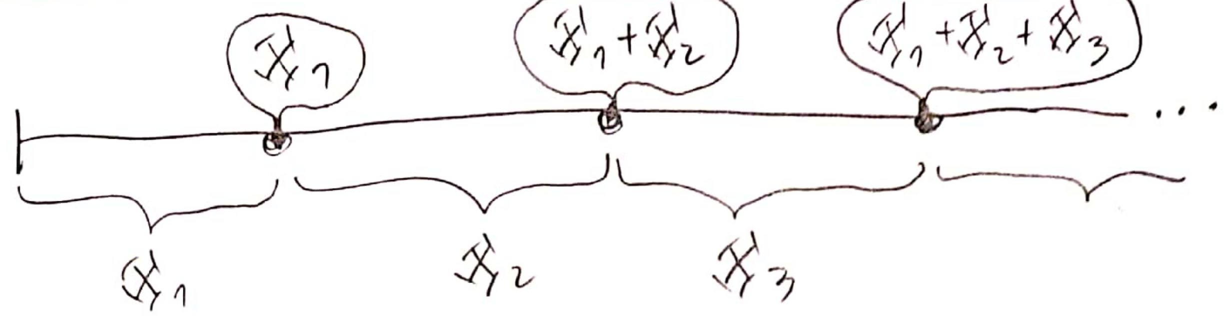
CONVOLUTION: IF X AND Y ARE INDEP. ABSOLUTELY CONTINUOUS R.V.'S AND $\boxed{Z_1 := X+Y}$ THEN $$f_{Z_1}(x) = \underbrace{\int_{-\infty}^{\infty} f_{X}(y)\cdot f_{Y}(x-y)\,dy}$$

$$\boxed{f_{Z_1} = f_{X} * f_{Y}} \leftarrow \text{THE CONVOLUTION OF } f_{X} \text{ AND } f_{Y}$$

$\boxed{10.2}$ a) $$f_{X_i}(x) = \lambda\cdot e^{-\lambda\cdot x}\cdot\mathbb{1}[x>0]$$

$$f_{Z_1}(x) = \int_{-\infty}^{\infty} \lambda\cdot e^{-\lambda\cdot y}\cdot\mathbb{1}[y>0]\cdot\lambda\cdot e^{-\lambda\cdot(x-y)}\cdot\mathbb{1}[x-y>0]\,dy =$$

$$= \int_0^x \lambda^2\cdot e^{-\lambda\cdot x}\,dy = \lambda^2\cdot x\cdot e^{-\lambda\cdot x}\cdot\mathbb{1}[x>0]$$



THESE ARE THE POINTS OF A POISSON POINT PROCESS WITH INTENSITY λ.

ALTERNATIVE SOLUTION OF a):

IF $F(x)$ IS THE C.D.F. OF $X_1 + X_2$

THEN $F(x) = \mathbb{P}($SECOND POINT $\leq x) =$

$\mathbb{P}($NUMBER OF POINTS IN $[0,x]$ IS ≥ 2

$$\mathbb{P}(\text{Poi}(\lambda \cdot x) \geq 2) = 1 - e^{-\lambda \cdot x} - \lambda \cdot x \cdot e^{-\lambda \cdot x}$$

P.D.F. OF SECOND POINT:

$$f(x) = F'(x) = \frac{d}{dx}\left(1 - e^{-\lambda \cdot x} - \lambda \cdot x \cdot e^{-\lambda \cdot x}\right) = \lambda^2 \cdot x \cdot e^{-\lambda x}$$

b) $\mathbb{P}(X_1 + X_2 + X_3 > t) = \mathbb{P}($THIRD POINT $> t) =$

$= \mathbb{P}($NUMBER OF POINTS IN $[0,t]$ IS $\leq 2) =$

$$= \sum_{n=0}^{2} e^{-\lambda \cdot t} \cdot \frac{(\lambda \cdot t)^n}{n!} = e^{-\lambda \cdot t} + e^{-\lambda \cdot t} \cdot \lambda t + e^{-\lambda \cdot t} \cdot \frac{(\lambda \cdot t)^2}{2!}$$

---

c) $\mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = \frac{3}{\lambda}$

$$\text{Var}(X_1 + X_2 + X_3) \underset{\text{INDEPENDENCE}}{=} \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = \frac{3}{\lambda^2}$$

10.4 a) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx =$

$$= \int_0^{\infty} \frac{1}{y} \exp\left(-\left(y+\frac{x}{y}\right)\right) dx = e^{-y} \cdot \underbrace{\int_0^{\infty} \frac{1}{y} \cdot e^{-\frac{1}{y}\cdot x}\,dx}_{=1} = e^{-y}$$

IF $y > 0$
OTHERWISE 0

SINCE $\frac{1}{y} \cdot e^{-\frac{1}{y}\cdot x}$ IS THE P.D.F. OF AN $\mathrm{EXP}\left(\frac{1}{y}\right)$ R.V.

THUS $Y \sim \mathrm{EXP}(1)$, THUS $\mathbb{E}(Y) = 1$

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b) CONDITIONAL PROBABILITY DENSITY FUNCTION OF X GIVEN $Y = y$: $\boxed{f_{X|Y}(x|y) := \frac{f(x,y)}{f_Y(y)}}$

IN OUR CASE: $f_{X|Y}(x|y) = \dfrac{(1/y) \cdot \exp\left(-\left(y + \frac{x}{y}\right)\right)}{e^{-y}} =$

$$= \frac{1}{y} \cdot e^{-\frac{1}{y}\cdot x} \cdot \mathbb{I}[x > 0]$$

THUS IF WE KNOW THAT $Y = y$ THEN THE CONDITIONAL DISTRIBUTION OF X IS $\mathrm{EXP}\left(\frac{1}{y}\right)$

THUS $\boxed{\mathbb{E}(X \mid Y = y) = \frac{1}{1/y} = y}$ $\boxed{\mathbb{E}(X|Y) = Y}$

TOWER RULE OF CONDITIONAL EXPECTATION:

$\boxed{\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))} = \mathbb{E}(Y) = 1$

$\boxed{10.6}$ $X_i := \mathbb{1}[A_i] := \begin{cases} 1 & \text{IF } A_i \text{ OCCURS} \\ 0 & \text{IF } A_i \text{ DOES NOT OCCUR} \end{cases}$

WHERE $A_i = \{ i\text{'TH GUEST SITS AT A NEW TABLE} \}$

NUMBER OF OCCUPIED TABLES $= X_1 + \dots + X_N$

$$\mathbb{E}(-\shortparallel-) \overset{\text{LIN}}{=\!=} \sum_{i=1}^{N} \underbrace{\mathbb{E}(X_i)}_{\mathbb{P}(A_i)} = ✪$$

$$\mathbb{P}(A_i) = \mathbb{P}\left(\begin{array}{c} i\text{'TH GUEST IS NOT} \\ \text{A FRIEND OF GUESTS} \\ 1, 2, \dots, i-1 \end{array} \right) = (1-p)^{i-1}$$

$$✪ = \sum_{i=1}^{N} (1-p)^{i-1} = \frac{1-(1-p)^N}{p}$$

$\boxed{10.7}$ $A_i := \{ i\text{'TH COUPLE SIT NEXT TO EACH OTHER} \}$

$X_i := \mathbb{1}[A_i]$ $\quad X = X_1 + \dots + X_{10}$ $\quad \boxed{\text{SYMM.}}$

a) $\mathbb{E}(X) \overset{\text{LIN.}}{=\!=} \sum_{i=1}^{10} \mathbb{E}(X_i) = \sum_{i=1}^{10} \mathbb{P}(A_i) \overset{\text{SYMM.}}{=} 10 \cdot \underbrace{\mathbb{P}(A_1)}_{\frac{2}{19}}$

b) $Var(X) = \mathbb{E}(X^2) - \underbrace{\mathbb{E}(X)^2}_{\left(\frac{20}{19}\right)^2}$

$$\mathbb{E}(X^2) = \mathbb{E}\left(\left(\sum_{i=1}^{10} X_i\right)^2\right) = \mathbb{E}\left(\sum_{i,j=1}^{10} X_i \cdot X_j\right) =$$

$$\sum_{i,j=1}^{10} \mathbb{E}(X_i \cdot X_j) = \sum_{i,j=1}^{10} \mathbb{P}(A_i \cap A_j) = \ddot{\smile}$$

$$\boxed{X_i \cdot X_j = \mathbb{1}[A_i] \cdot \mathbb{1}[A_j] = \mathbb{1}[A_i \cap A_j]}$$

$$\ddot{\smile} = 10 \cdot \underbrace{\mathbb{P}(A_1 \cap A_1)}_{= \mathbb{P}(A_1) = \frac{2}{19}} + 90 \cdot \underbrace{\mathbb{P}(A_1 \cap A_2)}_{\ddot{\frown}}$$

$$\ddot{\frown} = \frac{2 \cdot 2 \cdot 17!}{19!} = \frac{4}{19 \cdot 18}$$

BECAUSE THERE ARE 19! WAYS TO SEAT THEM IF WE DON'T DISTINGUISH SEATING ARRANGEMENTS THAT ARE OBTAINED FROM EACH OTHER USING A ROTATION, AND THERE ARE 17! WAYS TO SEAT THEM IF JOE AND JACK ARE BOTH GLUED TO THEIR WIVES, AND THERE ARE $2 \cdot 2 = 4$ WAYS TO UNGLUE THEM.

$\boxed{10.8}$ MONDAY: $Y' \sim N(7{:}30, 3^2)$, $Z_1 \sim N(7.37, 4^2)$

$\boxed{\text{I MISS THE CONNECTION}} \iff \boxed{Y' + 2 > Z_1} \iff \boxed{Z_1 - Y' < 2}$

Z_1 AND Y' ARE INDEP. WITH NORMAL DISTR.,

THUS IF $X' := Z_1 - Y'$ THEN

$$X' \sim N(7, \underbrace{3^2 + 4^2}_{\text{SUM OF VARIANCES}}) \sim N(7, 5^2)$$

DIFFERENCE OF EXPECTATIONS

$\boxed{\text{I MISS MY CONNECTION}} \iff \boxed{X' < 2}$

$$P(X' < 2) = P\left(\frac{X' - 7}{5} < \frac{2-7}{5}\right) = \Phi(-1) =$$

$$= 1 - \Phi(1) = 1 - 0.8413 = 0.1587 =: p$$

a) NUMBER OF TIMES I MISS MY CONNECTION THIS WEEK : $BIN(5, p)$

$$P(\text{—''—} \leq 1) = \binom{5}{0} \cdot p^0 \cdot (1-p)^5 + \binom{5}{1} \cdot p^1 \cdot (1-p)^4 =$$

$$= 0.819$$

---

b) MONDAY RETURN TRIP: $X^* \sim N(9, 5^2)$

$$P(X'^* < 2) = 1 - \Phi(1.4) \approx 0.0808 = p^*$$

NUMBER OF YEARLY MISSES MORNING: $V \sim BIN(220, p)$

—''— AFTERNOON: $V^* \sim BIN(220, p^*)$

DE MOIVRE – LAPLACE:

$$\boxed{\underbrace{\frac{V-220\cdot p}{\sqrt{220}}}_{W} \sim \mathcal{N}(0,\ p\cdot(1-p))} \qquad \underbrace{\frac{V^*-220\cdot p^*}{\sqrt{220}}}_{W^*} \sim \mathcal{N}(0, p^*\cdot(1-p^*))$$

APPROXIMATELY

$\mathbb{P}$(I MISS MORE AFTERNOON CONNECTIONS THAN MORNING CONNECTIONS THIS YEAR) =

$$\mathbb{P}(V^* > V) = \mathbb{P}\left(V^* - 220\cdot p^* > V - 220\cdot p + 220\cdot(p-p^*)\right)$$

$$= \mathbb{P}\left(W^* > W + \frac{220\cdot(p-p^*)}{\sqrt{220}}\right) =$$

$$= \mathbb{P}(W^* - W > 1.155) =$$

$$= 1 - \Phi\left(\frac{1.155}{0.455}\right) = 0.0057$$

$W^* - W \sim \mathcal{N}(0, \sigma^2)$

$\sigma^2 = p\cdot(1-p) + p^*\cdot(1-p^*)$

THUS

$\sigma = 0.455$