Probability 1 – Exercises

Tutorial no. 11 14th Nov 2024

11.1 The random variables X and Y have a jointly absolutely continuous distribution, the marginal density function of X is $f_X(x) = \lambda^2 x e^{-\lambda x}$ if $x > 0$ and 0 if $x < 0$. The conditional distribution of Y given the value of X is uniform on the interval $[0, X]$. Calculate

(a) the joint probability density function of X and Y,

(b) the marginal density function of Y and the expectation $\mathbb{E}(Y)$,

(c) $\mathbb{E}(Y \mid X = x)$ and $\mathbb{E}(X \mid Y = y)$,

(d) $\mathbb{E}(X)$ (at this point it is simpler if we use the previous sub-exercise), and

(e) $\mathrm{Cov}(X, Y)$.

11.2 We throw a coin 10 times, with a probability p of heads and probability $(1 - p)$ of tails at each throw. Let X be the number of *pure runs* (e.g. in TTHTT, $X = 3$). Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$. (Hint: write X as a sum of some indicator variables of some pretty simple events.)

11.3 We throw with a die n times. Let X be the number of times we throw 1, and Y be the number of times we throw 2. What is the correlation of the two random variables?

HW 11.4 (3 points) There are 37 pockets on the roulette wheel, from 0 to 36. Xavier always bets that the result is at least 19. Yvette always bets that the result is 1 mod 3 (so, $1, 4, 7, \dots, 34$). Let's spin the wheel 20 times, independently. Let X be the number of times Xavier wins, and Y be the number of times Yvette wins. (It's possible that they both win a round, or that neither wins a round). What is the correlation of X and Y?

HW 11.5 (3 points) In an urn, there are M red and N blue balls. We take out all balls without replacement. Let X be the number of *pure runs.* (Similarly to question 11.1, but with red and blue instead of heads and tails). Calculate $\mathbb{E}(X)$.

11.6 Twelve people get on an elevator. They all choose a destination independently out of the 10 possible floors. Determine the expected value and the variance of the number of floors where the elevator stops.

11.7 Let (X, Y) has jointly uniform distribution on the triangle determined by the points $(-1, 0)$, $(0, 0)$, $(0, 2)$. (a) What is the two dimensional covariance matrix of (X, Y)?

(b) Let $Z = 7X + 2Y$. What is the two dimensional covariance matrix of (X, Z)?

HW 11.8 (4 points) Let X and Y be random variables that can only take two values: $X \in \{x_1, x_2\}$ and $Y \in \{y_1, y_2\}$. Prove that if $\mathrm{Cov}(X, Y) = 0$, then they are independent. (Not true in general!)

11.9 Let (X, Y) be the coordinates of a uniformly chosen point on the circumference of the circle centered at $(1, 1)$ with radius 1. $\mathrm{Cov}(X, Y) =?$