$\boxed{10.10}$ STANDARDIZE: $Z_1 := Y/\sqrt{3}$

THEN X_1, Z_1 ARE I.I.D. $N(0,1)$, THUS THE RANDOM VECTOR (X_1, Z_1) HAS 2-DIM STANDARD NORMAL DISTRIBUTION.

a) $\mathbb{P}(0 \le X \le 2, |Y| \le 1) =$

$\mathbb{P}(0 \le X \le 2, |\sqrt{3} \cdot Z_1| \le 1) \overset{\text{INDEP.}}{=}$

$\mathbb{P}(0 \le X \le 2) \cdot \mathbb{P}(-\frac{1}{\sqrt{3}} \le Z_1 \le \frac{1}{\sqrt{3}}) =$

$(\Phi(2) - \Phi(0)) \cdot (\Phi(\frac{1}{\sqrt{3}}) - \Phi(-\frac{1}{\sqrt{3}}))$

$= (\Phi(2) - \frac{1}{2}) \cdot (2 \cdot \Phi(\frac{1}{\sqrt{3}}) - 1)$

---

b) $\mathbb{P}(X + Y < 3) = ?$

$W := X + Y \quad W \sim N(0, 1^2 + \sqrt{3}^2) \sim N(0, 2^2)$

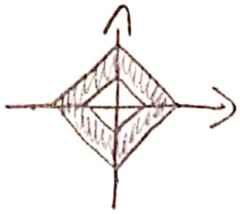
$? = \mathbb{P}(W < 3) = \mathbb{P}(\frac{W}{2} < \frac{3}{2}) = \Phi(\frac{3}{2})$

---

c) $\mathbb{P}(1 \le \sqrt{3} \cdot |X| + |Y| \le 3) =$

$\mathbb{P}(1 \le \sqrt{3} \cdot |X| + \sqrt{3} \cdot |Z_1| \le 3) =$

$\mathbb{P}(\frac{1}{\sqrt{3}} \le |X| + |Z_1| \le \sqrt{3}) = ☆$

$\circledast = \mathbb{P}\big((X, Z_1) \in$  $\big) = \dots$

USE THAT 2-DIM STANDARD NORMAL DISTRIBUTION IS INVARIANT UNDER ROTATIONS AROUND THE ORIGIN.

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d) $\mathbb{P}(1 \le 3\cdot X^2 + Y^2 \le 3) = \mathbb{P}(1 \le 3\cdot X^2 + 3\cdot Z_1^2 \le 3)$

$= \mathbb{P}\left(\frac{1}{3} \le X^2 + Z_1^2 \le 1\right) = \iint\limits_{\frac{1}{3} \le x^2+z^2 \le 1} f(x,z)\,dx\,dz = \dots$

WHERE $f(x,z) = \varphi(x)\cdot\varphi(z) = \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-z^2/2} =$

$= \frac{1}{2\pi}\cdot \exp\left(-\frac{1}{2}\cdot(x^2+z^2)\right)$ SWITCH TO POLAR COORDINATES

$\boxed{11.2}$ $B_i := \{$ i'TH AND (i+1)'ST COINS DIFFER $\}$

$Y_i := \mathbb{1}[B_i]$, $Y := \sum_{i=1}^{9} Y_i$, THEN

$X = Y + 1$ (NUMBER OF SWITCHES PLUS ONE)

$\mathbb{E}(X) \overset{\text{LIN.}}{=} 1 + \sum_{i=1}^{9} \underbrace{\mathbb{E}(Y_i)}_{\mathbb{P}(B_i)} = 1 + 9\cdot\underbrace{\left(p\cdot(1-p) + (1-p)\cdot p\right)}_{q := \mathbb{P}(B_i)}$

T.B.C.

PAGE 2

$$\mathrm{Var}(X) = \mathrm{Var}(Y+1) = \mathrm{Var}(Y) = \mathrm{Cov}(Y, Y) =$$

$$= \mathrm{Cov}\left(\sum_{i=1}^{9} Y_i, \sum_{j=1}^{9} Y_j\right) \overset{\text{BILINEARITY}}{=}$$

$$= \sum_{i,j=1}^{9} \mathrm{Cov}(Y_i, Y_j) = ✪$$

$\boxed{i=j}$: $\mathrm{Cov}(Y_i, Y_i) = \mathrm{Var}(Y_i) = q \cdot (1-q)$

$\boxed{|i-j| \geq 2}$: $\mathrm{Cov}(Y_i, Y_j) = 0$ BECAUSE Y_i AND Y_j ARE INDEPENDENT BECAUSE Y_i AND Y_j DEPEND ON SEPARATE COINS

$\boxed{|i-j| = 1}$ $\mathrm{Cov}(Y_i, Y_{i+1}) \overset{\boxed{\text{SYMM}}}{=} \mathrm{Cov}(Y_{i+1}, Y_i)$ AND THUS W.L.O.G. $j = i+1$

$$\mathrm{Cov}(Y_i, Y_{i+1}) = \mathbb{E}(Y_i \cdot Y_{i+1}) - \underbrace{\mathbb{E}(Y_i)}_{q} \cdot \underbrace{\mathbb{E}(Y_{i+1})}_{q}$$

$$\mathbb{E}(Y_i \cdot Y_{i+1}) = \mathbb{E}(\mathbb{1}[B_i] \cdot \mathbb{1}[B_{i+1}]) = \mathbb{E}(\mathbb{1}[B_i \cap B_{i+1}]) =$$

$$= \mathbb{P}(B_i \cap B_{i+1}) = p \cdot (1-p) \cdot p + (1-p) \cdot p \cdot (1-p) =: \tilde{q}$$

$$✪ = \underbrace{9 \cdot q \cdot (1-q)}_{\text{DIAGONAL TERMS}} + \underbrace{2 \cdot 8 \cdot (\tilde{q} - q^2)}_{\text{NEAR-DIAG. TERMS}}$$

$\boxed{11.3}$ $$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}$$

$X_i := \mathbb{1}[A_i]$, $A_i := \{$ i'TH DIE ROLL IS 1 $\}$

$Y_i := \mathbb{1}[B_i]$, $B_i := \{$ i'TH —"— IS 2 $\}$

$X = \sum_{i=1}^{n} X_i$, $Y = \sum_{i=1}^{n} Y_i$, $X \sim \mathrm{BIN}\left(n, \frac{1}{6}\right) \sim Y$

$$\mathrm{Var}(X) = n \cdot \frac{1}{6} \cdot \left(1 - \frac{1}{6}\right) = \mathrm{Var}(Y)$$

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) \overset{\boxed{\text{BILINEARITY}}}{=} \sum_{i,j=1}^{n} \mathrm{Cov}(X_i, Y_j)$$

$= ☺$ IF $\boxed{i \neq j}$ THEN $\mathrm{Cov}(X_i, X_j) = 0$ SINCE THEY ARE INDEP.

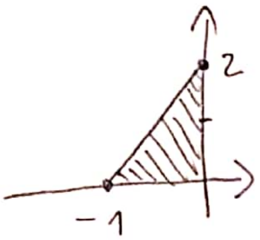
IF $\boxed{i = j}$ THEN $$\mathrm{Cov}(X_i, Y_i) = \mathbb{E}(X_i \cdot Y_i) - \underbrace{\mathbb{E}(X_i)}_{1/6} \cdot \underbrace{\mathbb{E}(Y_i)}_{1/6}$$

$\mathbb{E}(X_i \cdot Y_i) = \mathbb{P}(A_i \cap B_i) = 0$, THUS

$$☺ = \sum_{i=1}^{n} \mathrm{Cov}(X_i, X_j) = \frac{-n}{36}$$

$$\mathrm{Corr}(X, Y) = \frac{-n/36}{\sqrt{n \cdot \frac{5}{36} \cdot n \cdot \frac{5}{36}}} = -\frac{1}{5}$$: NEGATIVE CORRELATION: X AND Y WORK AGAINST EACH OTHER

$\boxed{11.7}$



COV. MATRIX:

$Var(X)$	$Cov(X,Y)$
$Cov(X,Y)$	$Var(Y)$

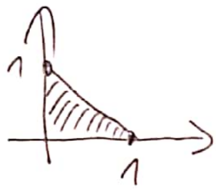
$=$

a	b
b	c

INSTEAD, LET $(\tilde{X}, \tilde{Y})$

BE UNIFORM ON

THIS TRIANGLE $\rightarrow$



$\tilde{a}$	$\tilde{b}$
$\tilde{b}$	$\tilde{c}$

AND LET US FIND THE COV. MATRIX

$$f(x,y) = 2 \cdot \mathbb{1}[0<x,\ 0<y,\ x+y<1]$$

$$\mathbb{E}(\tilde{X}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} x \cdot 2\,dx\,dy =$$

$$= \int_0^1 (1-y)^2\,dy = \frac{1}{3} = \mathbb{E}(\tilde{Y}) \quad \text{SYMM.}$$

$$\mathbb{E}(\tilde{X}^2) = \int_0^1\int_0^{1-y} x^2 \cdot 2\,dx\,dy = \frac{2}{3}\cdot\int_0^1 (1-y)^3\,dy = \frac{2}{12}$$

$$Var(\tilde{X}) = \frac{2}{12} - \left(\frac{1}{3}\right)^2 = \tilde{a} \stackrel{\boxed{\text{SYMM}}}{=} Var(\tilde{Y}) = \tilde{c}$$

$$\mathbb{E}(\tilde{X}\cdot\tilde{Y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} x \cdot y \cdot 2\,dx\,dy =$$

$$= \int_0^1 y\cdot(1-y)^2\,dy = \frac{1}{12}, \quad Cov(\tilde{X},\tilde{Y}) = \frac{1}{12} - \frac{1}{3}\cdot\frac{1}{3} = \tilde{b}$$

$$\boxed{X = -\tilde{X}} \quad \boxed{Y = 2\cdot\tilde{Y}} \quad \boxed{a = \tilde{a}} \quad \boxed{c = 2^2\cdot\tilde{c}}$$

$$b = Cov(-\tilde{X}, 2\cdot\tilde{Y}) \stackrel{\text{BIL.}}{=\!=} -2\cdot\tilde{b}$$

b) $\mathrm{Cov}(X, Z_1) = \mathrm{Cov}(X, 7X + 2Y) \overset{\leftarrow \text{BIL.}}{=}$

$= 7 \cdot \underbrace{\mathrm{Cov}(X, X)}_{a} + 2 \cdot \underbrace{\mathrm{Cov}(X, Y)}_{b}$

$\mathrm{Var}(Z_1, Z_1) = \mathrm{Cov}(Z_1, Z_1) =$

$\mathrm{Cov}(7X + 2Y, 7X + 2Y) \overset{\leftarrow \text{BILINEARITY}}{=}$

$= 49 \cdot \mathrm{Cov}(X, X) + 14 \cdot \mathrm{Cov}(X, Y) + 14 \cdot \mathrm{Cov}(Y, X) +$

$+ 4 \cdot \mathrm{Cov}(Y, Y) = 49 \cdot a + 28 \cdot b + 4 \cdot c$

NOTE: JOINT DISTRIBUTION OF (X, Z_1) IS UNIFORM ON THIS TRIANGLE: