Probability 1 – Exercises

Tutorial no. 12 — 28th Nov 2024

12.1 Let X and Y be two absolutely continuous random variables. Let X have a marginal PDF of $f_X(x) = \lambda^2 x \exp(-\lambda x)$ if $x \geq 0$, and 0 otherwise. Conditional on $X = x$, let Y be uniform on the interval $[0, x]$. What is:

(a) the joint PDF of (X, Y)?

(b) the marginal PDF of Y, and $\mathbb{E}(Y)$?

(c) $\mathbb{E}(X \mid Y = y)$ and $\mathbb{E}(Y \mid X = x)$?

(d) $\mathbb{E}(X)$? (Hint: use previous results)

(e) $\mathrm{Cov}(X, Y)$?

12.2 Let $X_1, \ldots, X_n$ be independent and identically distributed random variables (iid).

What is $\mathbb{E}(X_1 \mid X_1 + \cdots + X_n)$?

HW **12.3** (2 points) Let X and Y be i.i.d. (independent and identically distributed) positive random variables.

$$\mathbb{E}\left(\frac{X}{X+Y}\right) = ?$$

12.4 Let Y be a normal variable with mean μ and variance 1, and the conditional distribution of X given $Y = y$ is normal variable with mean y and variance σ^2. What is the conditional distribution of Y given $X = x$?

Interpretation: Y is the original signal, X is the observed noisy version of the signal (i.e., the additive noise was an independent normal variable with mean zero and variance σ^2) and after detecting that value of the noisy version, we want to infer the original signal.

HW **12.5** (3 points) We take a PQ line segment with length 3 and break it into two parts, at a uniformly chosen point R. Let X denote the length of the PR line segment. The other, $3 - X$ long line segment is broken again, at a uniformly chosen point S. Let Y denote the length of the RS line segment.

(a) What is the joint PDF of (X, Y)?

(b) Find $\mathbb{E}(Y \mid X = x)$.

(c) Calculate $\mathrm{Cov}(X, Y)$.

♣ **12.6** (3 points) Let X, Y and Z be independent random variables. Let the CDF of X and Y be $F(x)$ and $G(y)$, respectively. Let Z be a Bernoulli random variable with parameter p. Determine the CDF of the following distributions:

(a) $T := ZX + (1 - Z)Y$,

(b) $U := ZX + (1 - Z)\max\{X, Y\}$,

(b) $U := ZX + (1 - Z)\min\{X, Y\}$.

12.7 We roll a die. If the result is i, then we choose Y uniformly on $[0, i]$. What is $\mathbb{E}(Y)$ and $\mathbb{D}^2(Y)$?

12.8 The amount of time until a lightbulb breaks follows an exponential distribution with 9 months expected time. If the lightbulb breaks, I immediately replace it with the same kind of lightbulb. I use my lamp for T years, where T is a number uniformly chosen between 0 and 9 years. Let Y be the number of times I have to change the lightbulb during this time. $\mathbb{E}(Y) = ?$ and $\mathbb{D}^2(Y) = ?$

12.9 Let U_1 and U_2 be two independent random variables with uniform distribution on $[0,1]$. Show that if

$$X = \sqrt{-2\ln U_1} \cdot \cos(2\pi U_2)$$
$$Y = \sqrt{-2\ln U_1} \cdot \sin(2\pi U_2)$$

Then (X, Y) have a joint Normal distribution.

HW 12.10 (3 points) Let X and Y be the two coordinates of a point that we choose randomly on the disc with radius 1 around the origin. ($f(x,y) = \frac{1}{\pi}$ if $x^2 + y^2 \leq 1$.) What is the joint density function of $R = \sqrt{X^2 + Y^2}$ and $\Theta = \mathrm{arctg}(Y/X)$?

12.11 Let $X, Y \sim \mathrm{Exp}(\lambda)$ be independent. What is the joint density of $U := X + Y$ and $V := \frac{X}{X+Y}$? Show that U and V are independent.

HW 12.12 (2 points) Let $X_1, \ldots, X_n, X_{n+1}$ be independent, standard normal random variables, and let

$$Y = \frac{X_1 + \cdots + X_n}{n}.$$

(a) What is the covariance matrix of (X_1, Y)?

(b) Find $\mathbb{P}(|Y| \leq |X_{n+1}|)$.

12.13 Two rival research groups are trying to estimate the mass of a particle. The first group performs k measurements, while the second group performs l measurements. Both research groups estimate the mass of the particle by taking the empirical average of their respective measurements. The true mass of the particle is μ, and the (noisy) measurements have normal distribution with mean μ and variance of σ^2, independently of each other. What is the probability that the estimate of the first research group is better than that of the second research group?

12.14 Let X be a standard normal distribution, and I independent from X, with $\mathbb{P}(I = 1) = \mathbb{P}(I = 0) = \frac{1}{2}$. Define Y as

$$Y := \begin{cases} X, & \text{if } I = 1 \\ -X, & \text{if } I = 0. \end{cases}$$

(a) Show that Y is standard normal.

(b) Are I and Y independent?

(c) Are X and Y independent?

(d) Show that $\mathrm{Cov}(X, Y) = 0$.