$\boxed{12.1}$ $f_{Y|X}(y|x) = \frac{1}{x} \cdot \mathbb{1}[0<y<x]$

a) $f(x,y) = f_X(x) \cdot f_{Y|X}(y|x) =$

$= \lambda^2 \cdot x \cdot e^{-\lambda x} \cdot \mathbb{1}[x>0] \cdot \frac{1}{x} \cdot \mathbb{1}[0<y<x]$

$= \lambda^2 \cdot e^{-\lambda \cdot x} \cdot \mathbb{1}[0<y<x]$

---

b) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} \lambda^2 \cdot e^{-\lambda x} \cdot \mathbb{1}[0<y<x]\,dx =$

$= \int_y^{\infty} \lambda^2 \cdot e^{-\lambda x} \cdot \mathbb{1}[0<y]\,dx = \mathbb{1}[0<y] \cdot \lambda \cdot \int_y^{\infty} \lambda \cdot e^{-\lambda x}\,dx =$

$= \lambda \cdot e^{-\lambda y} \cdot \mathbb{1}[0<y]$, THUS $Y \sim EXP(\lambda)$,

THUS $\mathbb{E}(Y) = \frac{1}{\lambda}$

---

c) $\mathbb{E}(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy =$

$= \int_{-\infty}^{\infty} y \cdot \frac{1}{x} \cdot \mathbb{1}[0<y<x]\,dy = \frac{1}{x} \cdot \int_0^x y\,dy = \frac{x}{2}$

AND OF COURSE: Y IS UNIFORM ON $[0, X]$,

SO $\mathbb{E}(Y|X=x) = \frac{x}{2}$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^2 \cdot e^{-\lambda x} \cdot \mathbb{1}[0<y<x]}{\lambda \cdot e^{-\lambda \cdot y} \cdot \mathbb{1}[0<y]} \overset{\text{IF } y>0}{=} (*)$

$$\star = \lambda \cdot e^{-\lambda \cdot (x-y)} \cdot \mathbb{1}[y < x] =: g_y(x)$$

IF $Z_1 \sim EXP(\lambda)$ THEN THE P.D.F. OF $Z_1 + y$ IS EXACTLY THIS $g_y(x)$.

THUS $\mathbb{E}(X \mid Y = y) = \int_{-\infty}^{\infty} x \cdot g_y(x)\, dx =$

$= \mathbb{E}(Z_1 + y) = \mathbb{E}(Z_1) + y = \frac{1}{\lambda} + y$

---

d) TOWER PROPERTY OF CONDITIONAL EXPECTATION: $\boxed{\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y))}$

WHERE $\mathbb{E}(X \mid Y) = \psi(Y)$, WHERE $\mathbb{E}(X \mid Y = y) = \psi(y)$. IN OUR CASE: $\psi(y) = \frac{1}{\lambda} + y$

THUS: $\mathbb{E}(X) = \mathbb{E}(\psi(Y)) = \mathbb{E}\left(\frac{1}{\lambda} + Y\right) = \frac{1}{\lambda} + \mathbb{E}(Y) = \frac{2}{\lambda}$

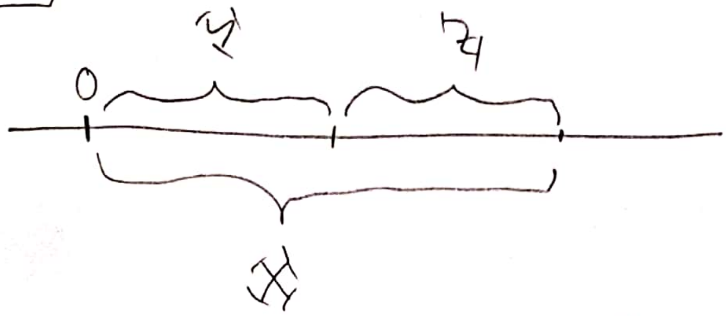
---

e) IF Y AND Z_1 ARE I.I.D. WITH $EXP(\lambda)$ DISTR. AND $X = Y + Z_1$ THEN $f(x,y) = \underbrace{f_Y(y)}_{\lambda \cdot e^{-\lambda \cdot y} \cdot \mathbb{1}[0<y]} \cdot \underbrace{f_{X|Y}(x \mid y)}_{g_y(x)}$

THIS IS THE SAME $f(x,y)$ AS IN (a), THUS

$$Cov(X, Y) = Cov(Y + Z_1, Y) \overset{\text{BILINEARITY}}{=} \underbrace{Cov(Y, Y)}_{\frac{1}{\lambda^2} = Var(Y)} + \underbrace{Cov(Z_1, Y)}_{\text{INDEP} \Rightarrow = 0}$$

$\boxed{12.1}$ MORAL OF THE STORY:



IF WE CONSIDER A POISSON PROC. WITH INTENSITY λ

THEN: Y IS THE TIME OF THE FIRST ARRIVAL

X IS THE TIME OF THE SECOND —''—

THUS: IF I KNOW THAT THE SECOND ARRIVAL OF THE P.P.P. HAPPENED AT $X = x$ THEN THE CONDITIONAL DISTRIBUTION OF THE FIRST ARRIVAL IS UNIFORM ON $[0, x]$.

THIS FACT IS THE CONTINUOUS ANALOGUE OF THE RESULT OF EXERCISE $\boxed{6.10}$.

TOWER PROPERTY: $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$

CONDITIONAL VARIANCE FORMULA:

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|Y)) + \mathrm{Var}(\mathbb{E}(X|Y))$$

$\boxed{12.7}$ $X \sim \mathrm{UNI}\{1,2,\dots,6\}$ (DIE ROLL)

$$f_{Y|X}(y \mid X=i) = \frac{1}{i}\cdot \mathbb{1}[0<y<i]$$

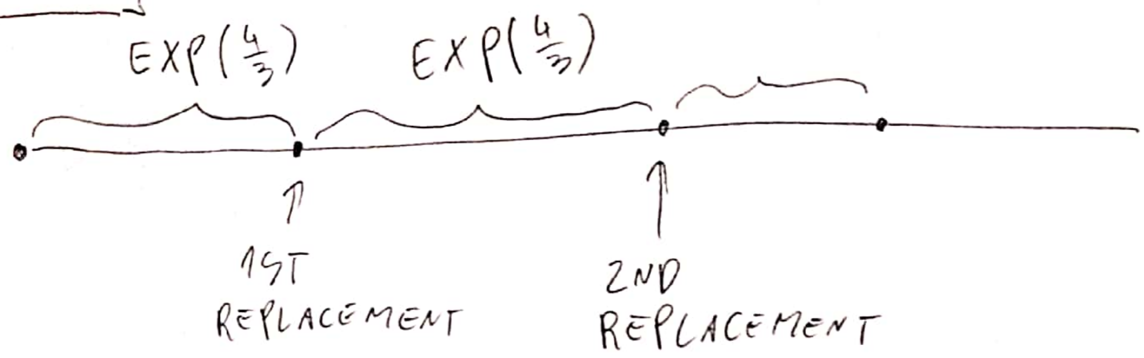
$\mathbb{E}(Y \mid X=i) = \frac{i}{2}$ (SINCE IT IS UNIFORM ON $[0,i]$)

$\mathbb{E}(Y|X) = \frac{X}{2}$, $\mathbb{E}(Y) = \mathbb{E}\left(\frac{X}{2}\right) = \frac{1}{2}\cdot\mathbb{E}(X) = \frac{1}{2}\cdot\frac{7}{2}$

$\mathrm{Var}(Y \mid X=i) = \frac{i^2}{12}$ (SINCE IT IS UNIF. ON $[0,i]$)

$$\mathrm{Var}(Y) = \underbrace{\mathbb{E}\left(\frac{X^2}{12}\right)}_{=\frac{1}{12}\cdot\underbrace{\frac{1^2+2^2+\dots+6^2}{6}}_{=\frac{91}{6}}} + \underbrace{\mathrm{Var}\left(\frac{X}{2}\right)}_{=\frac{1}{4}\cdot\underbrace{\mathrm{Var}(X)}_{\frac{91}{6}-\left(\frac{7}{2}\right)^2}}$$

$\boxed{12.8}$ OUR UNIT OF MEASUREMENT: YEAR



9 MONTHS $= \frac{3}{4}$ YEARS (EXPECTED VALUE OF $\mathrm{EXP}(\lambda)$ IS $\frac{1}{\lambda}$)

THUS THE TIMES OF REPLACEMENTS FORM A POISSON POINT PROC. WITH INTENSITY $4/3$.

THUS IF I USE THE LAMP FOR $T=t$ YEARS THEN THE NUMBER OF REPLACEMENTS Y WILL HAVE $\mathrm{POI}\left(\frac{4}{3}\cdot t\right)$ DISTRIBUTION.

I USE THE LAMP FOR $T \sim \mathrm{UNI}[0,9]$ YEARS.

$\mathbb{E}(Y \mid T=t) = \frac{4}{3}\cdot t$ (EXPECTED VALUE OF $\mathrm{POI}(\frac{4}{3}t)$)

$\mathbb{E}(Y \mid T) = \frac{4}{3}\cdot T$, $\mathbb{E}(Y) = \mathbb{E}\left(\frac{4}{3}T\right) = \frac{4}{3}\mathbb{E}(T) = \frac{4}{3}\cdot\frac{9}{2}$

$\mathrm{Var}(Y \mid T=t) = \frac{4}{3}\cdot t$ (VARIANCE OF $\mathrm{POI}(\frac{4}{3}t)$)

$$\mathrm{Var}(Y) = \underbrace{\mathbb{E}(\mathrm{Var}(Y \mid T))}_{\mathbb{E}\left(\frac{4}{3}T\right) = \frac{4}{3}\cdot\frac{9}{2}} + \underbrace{\mathrm{Var}\left(\frac{4}{3}T\right)}_{=\left(\frac{4}{3}\right)^2\cdot \mathrm{Var}(T)}, \quad \mathrm{Var}(T) = \frac{9^2}{12}$$

MULTI-DIM. DENSITY TRANSFORM:

IF $\begin{pmatrix} X \\ Y \end{pmatrix} = \underline{\Psi} \begin{pmatrix} U \\ V \end{pmatrix}$ $\left(\text{I.E.: } \begin{matrix} X = \Psi_1(U,V) \\ Y = \Psi_2(U,V) \end{matrix} \right)$

AND THE JOINT P.D.F. OF (U,V) IS $f(u,v)$

THEN THE JOINT P.D.F. OF (X,Y) IS:

$$g(x,y) = \sum_{\substack{(u,v) \in \\ \Psi^{-1}(x,y)}} f(u,v) \cdot \frac{1}{|\underline{\underline{J}}(u,v)|}$$

$$\boxed{\begin{pmatrix} U \\ V \end{pmatrix} = \underline{\Psi}^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}}$$

$\underline{\underline{J}}(u,v)$ DENOTES THE JACOBI MATRIX OF $\underline{\Psi}$

EVALUATED AT (u,v): $\underline{\underline{J}}(u,v) = \begin{array}{|c|c|} \hline \partial_u \Psi_1(u,v) & \partial_v \Psi_1(u,v) \\ \hline \partial_u \Psi_2(u,v) & \partial_v \Psi_2(u,v) \\ \hline \end{array}$

AND $|\underline{\underline{J}}(u,v)|$ DENOTES THE ABSOLUTE VALUE

OF THE DETERMINANT OF $\underline{\underline{J}}(u,v)$.

$\boxed{12.9}$ (X,Y) HAS 2D STANDARD NORMAL DISTR.
IF X AND Y ARE I.I.D. $\mathcal{N}(0,1)$

THUS THE JOINT P.D.F. OF (X,Y) IS:

$$g(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}}_{f_X(x)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2}}_{f_Y(y)} = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}\cdot(x^2+y^2)}$$

JOINT P.D.F. OF (U_1, U_2): $f(u,v) = \mathbb{1}[0 \le u \le 1, 0 \le v \le 1]$

$$x = \Psi_1(u,v) = \sqrt{-2 \cdot \ln(u)} \cdot \cos(2\pi v)$$
$$y = \Psi_2(u,v) = \sqrt{-2 \cdot \ln(u)} \cdot \sin(2\pi v)$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\Psi}\begin{pmatrix} u \\ v \end{pmatrix}}$$

POLAR: $\boxed{r = \sqrt{x^2+y^2} = \sqrt{-2\ln(u)}}$ $\boxed{\varphi = 2\pi v}$

(THUS $2\pi \cdot U_2$ IS THE <u>ANGLE</u> OF THE PLANAR VECTOR (X, Y), $\sqrt{-2 \cdot \ln(U_1)}$ IS ITS <u>LENGTH</u>)

<u>THUS</u>: <u>LENGTH</u> AND <u>ANGLE</u> ARE INDEPENDENT!

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} = \underline{\Psi}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}}$$

$$u = \Psi_1^{-1}(x,y) = e^{-\frac{1}{2}r^2} = \exp\left(-\frac{1}{2} \cdot (x^2+y^2)\right)$$
$$v = \Psi_2^{-1}(x,y) = \frac{1}{2\pi} \cdot \arg(x + i \cdot y)$$

$$\underline{\underline{J}}(u,v) = \begin{bmatrix} \frac{-1}{\sqrt{-2 \cdot \ln(u)} \cdot u} \cdot \cos(2\pi v) & \sqrt{-2 \cdot \ln(u)} \cdot 2\pi \cdot (-\sin(2\pi v)) \\ \frac{-1}{\sqrt{-2 \cdot \ln(u)} \cdot u} \cdot \sin(2\pi v) & \sqrt{-2 \cdot \ln(u)} \cdot 2\pi \cdot \cos(2\pi v) \end{bmatrix}$$

$$\det\left(\underline{\underline{J}}(u,v)\right) = \frac{-2\pi}{u} \cdot \cos^2(2\pi v) - \frac{-2\pi}{u} \cdot \left(-\sin^2(2\pi v)\right) = \frac{-2\pi}{u}$$

THUS $g(x,y) = \sum_{\substack{(u,v) \in \\ \underline{\Psi}^{-1}(x,y)}} \underbrace{f(u,v)}_{1} \cdot \frac{1}{|\underline{\underline{J}}(u,v)|} =$

$\boxed{\underline{\Psi} \text{ IS INJECTIVE}}$

$$= \frac{1}{|\underline{\underline{J}}(u,v)|} = \frac{u}{2\pi} = \frac{1}{2\pi} \cdot \exp\left(-\frac{1}{2} \cdot (x^2+y^2)\right)$$

$\boxed{12.13}$ $X_i \sim N(\mu, \sigma^2)$, $1 \le i \le r$

$Y_i \sim N(\mu, \sigma^2)$, $1 \le j \le l$

$$X = \frac{1}{r} \sum_{i=1}^{r} X_i, \quad X \sim N\left(\frac{r \cdot \mu}{r}, \frac{r \cdot \sigma^2}{r^2}\right) \sim N\left(\mu, \frac{\sigma^2}{r}\right)$$

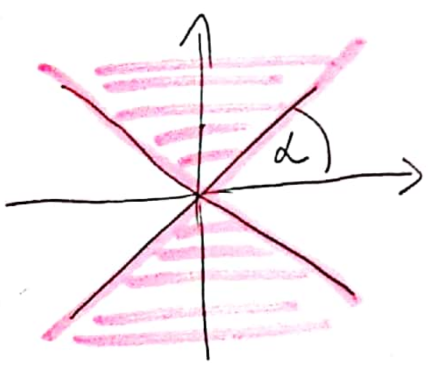
$$Y = \frac{1}{l} \cdot \sum_{j=1}^{r} Y_j, \quad Y \sim N\left(\mu, \frac{\sigma^2}{r}\right)$$

$$\mathbb{P}(|X - \mu| \le |Y - \mu|) = ?$$

STANDARDIZE: $X^* = \frac{X - \mu}{\sigma} \cdot \sqrt{r}$, $Y^* = \frac{Y - \mu}{\sigma} \cdot \sqrt{l}$

THUS X^*, Y^* ARE I.I.D. $N(0,1)$

$$? = \mathbb{P}\left(\frac{|X^*|}{\sqrt{r}} \le \frac{|Y^*|}{\sqrt{l}}\right) = \mathbb{P}\left((X^*, Y^*) \in \text{PINK}\right)$$



THE SLOPES OF THE PINK LINES ARE $\pm\sqrt{\frac{l}{r}}$

LET $\boxed{\alpha := \arctan\left(\sqrt{\frac{l}{r}}\right)}$

$\boxed{12.9}$ $\Longrightarrow$ ANGLE OF (X^*, Y^*) IS UNI $[0, 2\pi]$

$$\text{THUS } ? = \frac{\text{TOTAL FAVORABLE ANGLE}}{\text{TOTAL ANGLE}} =$$

$$= \frac{4 \cdot (\pi/2 - \alpha)}{2\pi} = 1 - \frac{2\alpha}{\pi}$$