Probability 1 – Exercises

Tutorial no. 13 — 5th Dec 2023

Central Limit Theorem (C.L.T.): If $X_1, X_2, \ldots$ are independent and identically distributed random variables, $\mathbb{E}(X_i) = \mu$, $\mathrm{Var}(X_i) = \sigma^2$, and $S_n = X_1 + \cdots + X_n$, then for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\cdot\mu}{\sqrt{n}\cdot\sigma} < x\right) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,\mathrm{d}y.$$

(Special case: if $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = 0) = 1 - p$, then $S_n \sim \mathrm{BIN}(n, p)$, and we get back the *de Moivre-Laplace theorem*)

13.1 Determine the moment generating function of the $\mathrm{GEO}(p)$ and $\mathrm{UNI}[a, b]$ distributions, and calculate their expected value and standard deviation based on this!

13.2 We are given a hundred lamps, whose lifetimes are independent exponential random variables with an expected value of 5 hours. We use the lamps one after the other, immediately replacing the one that burns out. Estimate the probability that there is still a working lamp after 525 hours.

13.3 The expected value of the students' scores on the probability exam is 74 and the standard deviation is 14. The first exam is written by 25 students, while the second exam is written by 64 students. Estimate the probability that

(a) the average score on the first exam is at least 80.

(b) the average score on the second exam is better than the first.

(c) the average score on the second exam is at least 2.2 points better than the first.

(d) the average score on the second exam is closer to 74 points than that of the first exam.

13.4 An astronomer measures the distance to a celestial body μ light years away. The results of her measurements are i.i.d. random variables with expected value μ and a standard deviation of 2 light years. How many times should she repeat her measurement so that the empirical mean is closer than 0.5 light years to μ with at least 95% probability?

13.5 We toss a fair coin 60 times. Denote the number of heads by X.

(a) Estimate the probability $\mathbb{P}(|X - 30| \geq 20)$ using Chebyshev's inequality.

(b) Why is it not a good idea to estimate $\mathbb{P}(|X - 30| \geq 20)$ using the de Moivre-Laplace theorem?

(c) Let $Y_\beta := \exp(\beta X)$. $\mathbb{E}(Y_\beta) =$?

(d) $\mathbb{P}(X \geq 50) \leq$? (if we apply Markov's inequality to Y_β)

(e) Find the value of the β that gives the best upper bound.

(f) $\mathbb{P}(|X - 30| \geq 20) \leq$?