13.1 MOMENT GENERATING FUNCTION:

IF X IS A RANDOM VARIABLE THEN $M(t) := \mathbb{E}(e^{t \cdot X})$

FACTS: $M'(0) = \mathbb{E}(X)$ $\quad M''(0) = \mathbb{E}(X^2)$

13.1 $X \sim GEO(p)$: $\mathbb{P}(X = k) = p \cdot (1-p)^{k-1}$

$k = 1, 2, \dots$

L.O.T.U.S.

$$\mathbb{E}(e^{t \cdot X}) = \sum_{k=1}^{\infty} e^{t \cdot k} \cdot \mathbb{P}(X = k) = \sum_{k=1}^{\infty} e^{t \cdot k} \cdot p \cdot (1-p)^{k-1} =$$

$$= p \cdot e^{t} \cdot \sum_{k=1}^{\infty} \left(e^{t} \cdot (1-p)\right)^{k-1} = \begin{cases} +\infty, & \text{IF } e^{t} \cdot (1-p) \geq 1 \\ \dfrac{p \cdot e^{t}}{1 - e^{t} \cdot (1-p)}, & \text{IF } e^{t} \cdot (1-p) < 1 \end{cases}$$

THUS $M(t) = \dfrac{p}{e^{-t} - (1-p)}$ IF $t < \ln\left(\dfrac{1}{1-p}\right)$

$$M'(t) = \frac{p \cdot e^{-t}}{(p + e^{-t} - 1)^2} \qquad M''(t) = p \cdot \left(\frac{2 \cdot e^{-2t}}{(p + e^{-t} - 1)^3} - \frac{e^{-t}}{(p + e^{-t} - 1)^2}\right)$$

$$\mathbb{E}(X) = M'(0) = \frac{1}{p} \qquad Var(X) = M''(0) - \left(\frac{1}{p}\right)^2 =$$

$$= p \cdot \left(\frac{2}{p^3} - \frac{1}{p^2}\right) - \frac{1}{p^2} = \frac{1-p}{p^2} \checkmark$$

$X \sim UNI[a, b]$ $\quad f(x) = \frac{1}{b-a} \cdot \mathbb{1}[a < x < b]$

$$\mathbb{E}(e^{t \cdot X}) \overset{L.O.T.U.S}{=\!=} \int_{-\infty}^{\infty} e^{t \cdot x} \cdot f(x)\, dx = \int_a^b e^{t \cdot x} \cdot \frac{1}{b-a} \underset{\uparrow}{=}$$

IF $t \neq 0$

$$= \frac{e^{t \cdot b} - e^{t \cdot a}}{t \cdot (b-a)} = M(t)$$

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$\lim_{t \to 0} M(t) = 1$ AND OF COURSE: $\mathbb{E}(e^{0 \cdot X}) = 1$ ✓

$$M'(t) = \ldots = \frac{(a \cdot t - 1) \cdot e^{a \cdot t} + (1 - b \cdot t) \cdot e^{b \cdot t}}{t^2 \cdot (a-b)}$$

$$\lim_{t \to 0} M'(t) \underset{\uparrow}{=\!=} \frac{a^2 - b^2}{2 \cdot (a-b)} = \frac{a+b}{2}$$ ✓

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<u>CENTRAL</u> <u>LIMIT</u> <u>THEOREM</u> <u>(C.L.T.)</u>:

IF $X_1, X_2, \ldots$ ARE I.I.D., $\mathbb{E}(X_i) = \mu$, $Var(X_i) = \sigma^2$,

$S_n = X_1 + \ldots + X_n$ THEN $\forall x \in \mathbb{R}$:

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{S_n - n \cdot \mu}{\sigma \cdot \sqrt{n}} < x \right) = \Phi(x)$$

C.D.F. OF $\mathcal{N}(0,1)$

AN APPLICATION: IF $X_i \sim POI(1)$ THEN

$S_n \sim POI(n)$ | $\mathbb{E}(X_i) = 1$ | $Var(X_i) = 1$

THUS $\mathbb{P}\left(\frac{S_n - n}{\sqrt{n}} < x\right) \overset{\text{C.L.T.}}{\approx} \Phi(x)$ (✪) IF n IS LARGE

13.2 $X_i \sim EXP\left(\frac{1}{5}\right)$ $\mu = 5$ $\sigma^2 = 25$ C.L.T.

$$\mathbb{P}(S_{100} > 525) = \mathbb{P}\left(\frac{S_{100} - 5\cdot 100}{\sqrt{25}\cdot\sqrt{100}} > \frac{525 - 5\cdot 100}{\sqrt{25}\cdot\sqrt{100}}\right) \approx$$

$Y \sim N(0,1)$

$$\approx \mathbb{P}\left(Y > \frac{25}{50}\right) = 1 - \Phi\left(\frac{1}{2}\right) = 1 - 0.6915 = 0.3085$$

ALTERNATIVE SOLUTION: IF WE HAD ∞-LY MANY LIGHTBULBS THEN THE TIMES OF LIGHTBULB CHANGES WOULD FORM A P.P.P. WITH INTENSITY $1/5$. THUS THE NUMBER N OF LIGHTBULB CHANGES IN $[0, 525]$ WOULD HAVE $POI(105)$ DISTRIBUTION.

$$\mathbb{P}(N < 100) = \mathbb{P}\left(\frac{N - 105}{\sqrt{105}} < \frac{100 - 105}{\sqrt{105}}\right) \overset{✪}{\approx}$$

$$\approx \Phi\left(\frac{-5}{\sqrt{105}}\right) = \Phi(-0.488) = 1 - \Phi(0.488) \approx$$

$$1 - 0.6879 = 0.3121$$

13.3 $\mathbb{E}(X_i) = 74 \quad \sqrt{Var(X_i)} = 14$

a) $\mathbb{P}\left(\frac{S_{25}}{25} \geq 80\right) = \mathbb{P}(S_{25} \geq 2000) =$

$$\mathbb{P}\left(\frac{S_{25} - 25\cdot 74}{\sqrt{25}\cdot 14} \geq \frac{2000 - 25\cdot 74}{\sqrt{25}\cdot 14}\right) \overset{C.L.T.}{\approx} \mathbb{P}\left(Y \geq \frac{150}{70}\right)$$

$$= 1 - \Phi\left(\frac{15}{7}\right) = 1 - 0.9838 = 0.0162$$

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b) $\frac{1}{2}$ AND WE WILL SEE WHY FROM THE SOLUTION OF c)

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c) $S_{25} = X_1 + \dots + X_{25}, \quad S^*_{64} = X^*_1 + \dots + X^*_{64}$

$$Y := \frac{S_{25} - 25\cdot 74}{5\cdot 14} \qquad Y^* = \frac{S^*_{64} - 64\cdot 74}{8\cdot 14}$$

$$\mathbb{P}\left(\frac{S_{25}}{25} + 2.2 < \frac{S^*_{64}}{64}\right) =$$

$$\mathbb{P}\left(\frac{5\cdot 14\cdot Y + 25\cdot 74}{25} + 2.2 < \frac{8\cdot 14\cdot Y^* + 64\cdot 74}{64}\right) =$$

$$\mathbb{P}\left(\frac{14}{5}\cdot Y + 2.2 < \frac{14}{8}\cdot Y^*\right) = \mathbb{P}\left(2.2 < \frac{14}{8}\cdot Y^* - \frac{14}{5}\cdot Y\right)$$ ☺

C.L.T. $\Rightarrow Y \approx \mathcal{N}(0,1), \quad Y^* \approx \mathcal{N}(0,1)$, I.I.D.

THUS $Z := \left(\frac{14}{8}\cdot Y^* - \frac{14}{5}\cdot Y\right) \sim \mathcal{N}\left(0, \left(\frac{14}{8}\right)^2 + \left(\frac{14}{5}\right)^2\right) = \mathcal{N}(0, \sigma^2)$

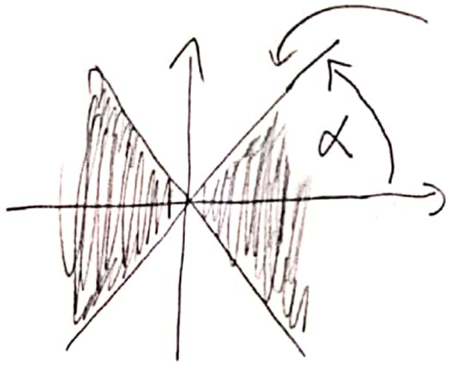
$\boxed{\sigma = 3.3}$ ☺ $= \mathbb{P}(2.2 < Z) = 1 - \Phi\left(\frac{2.2}{3.3}\right)$

13.3 (d) $\mathbb{P}\left(\left|\frac{S^*_{64}}{64}-74\right| < \left|\frac{S_{25}}{25}-74\right|\right) =$

$\mathbb{P}\left(\left|\frac{14}{8}\cdot Y^*\right| < \left|\frac{14}{5}\cdot Y\right|\right) = \mathbb{P}\left(|Y^*| < \frac{8}{5}\cdot |Y|\right) = \circledast$

(Y, Y^*) IS 2DIM STANDARD NORMAL

SLOPE OF LINE: $\frac{8}{5}$



$\circledast = \mathbb{P}\left((Y, Y^*) \in \text{SHADED}\right)$

$\alpha = \arctan\left(\frac{8}{5}\right) = 1.01$

ANGLE OF (Y, Y^*) IS UNIFORM ON $[0, 2\pi]$

$\circledast = \frac{4\alpha}{2\pi} = 0.64$

13.4 $\mathbb{E}(X_i) = \mu$, $\sqrt{Var(X_i)} = 2$

$0.95 \overset{\text{WANT}}{=\!=} \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \leq 0.5\right) = \mathbb{P}\left(\left|\frac{S_n - n\cdot\mu}{2\cdot\sqrt{n}}\right| \leq \frac{0.5\cdot\sqrt{n}}{2}\right)$

C.L.T.

$\approx \mathbb{P}\left(|Y| \leq \frac{\sqrt{n}}{4}\right) = \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right) =$

$= 2\cdot\Phi\left(\frac{\sqrt{n}}{4}\right) - 1$, THUS: WANT: $0.975 = \Phi\left(\frac{\sqrt{n}}{4}\right)$

WHERE $Y \sim N(0,1)$

$\frac{\sqrt{n}}{4} = \Phi^{-1}(0.975) = 1.96$, THUS $\boxed{n = 62}$

PAGE 5

13.5 $X \sim BIN(60, \frac{1}{2})$ $\mathbb{E}(X) = 30$, $Var(X) = 15$

a) $\mathbb{P}(|X - 30| \geq 20) \overset{\text{CHEB.}}{\leq} \frac{Var(X)}{(20)^2} = 0.0375$

b) DE MOIVRE-LAPLACE IS NOT TOO INFORMATIVE:

$\mathbb{P}\left(\left|\frac{X-30}{\sqrt{15}}\right| \geq \frac{20}{\sqrt{15}}\right) \approx \mathbb{P}(|Y| \geq 5.16) =$ (where $Y \sim \mathcal{N}(0,1)$)

$2 \cdot (1 - \underbrace{\Phi(5.16)}) \approx 2 \cdot (1-1) = 0$

THIS VALUE DOES NOT EVEN APPEAR ON OUR Φ TABLE, AND ALSO THE ERROR TERM OF THE C.L.T. APPROXIMATION IS OF ORDER $\frac{1}{\sqrt{n}}$ (ACCORDING TO THE BERRY-ESSÉN THEOREM), SO THE ERROR TERM IS BIGGER THAN THE PROBABILITY THAT WE WANT TO APPROX.

c) $\mathbb{E}(Y_\beta) = \underbrace{M_X(\beta)} = \ldots = 2^{-60} \cdot (1 + e^\beta)^{60}$

MOMENT GENERATING FUNCTION OF X EVALUATED AT β

d) $\mathbb{P}(X \geq 50) \overset{\beta \geq 0}{=} \mathbb{P}\left(e^{\beta \cdot X} \geq e^{\beta \cdot 50}\right) =$

$\mathbb{P}\left(Y_\beta \geq e^{\beta \cdot 50}\right) \overset{\text{MARKOV}}{\leq} \frac{\mathbb{E}(Y_\beta)}{e^{\beta \cdot 50}} = \frac{2^{-60} \cdot (1+e^\beta)^{60}}{e^{\beta \cdot 50}}$

e) $g(\beta) := \frac{2^{-60} \cdot (1+e^\beta)^{60}}{e^{\beta \cdot 50}}$ $\boxed{\min_{\beta > 0} g(\beta) = ?}$

$f(\beta) := \frac{1}{60} \cdot \ln(g(\beta)) = \ln\left(\frac{1+e^\beta}{2}\right) - \frac{5}{6}\beta$

$f'(\beta) = \frac{e^\beta}{1+e^\beta} - \frac{5}{6}$, $f''(\beta) > 0$ (f IS CONVEX)

$\boxed{f'(\beta^*) = 0} \Longrightarrow \boxed{\beta^* = \ln(5)}$ $g(\beta^*) = \frac{3^{60}}{5^{50}}$

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e) $\mathbb{P}(|X - 30| \geq 20) = \underbrace{\mathbb{P}(X \geq 50)}_{\leq \frac{3^{60}}{5^{50}}} + \underbrace{\mathbb{P}(X \leq 10)}_{☺}$

☺ $= \mathbb{P}$(NUMBER OF TAILS $\geq 50) \leq \frac{3^{60}}{5^{50}}$, THUS

$\mathbb{P}(|X - 30| \geq 20) \leq 2 \cdot \frac{3^{60}}{5^{50}} < 10^{-6}$ (MUCH BETTER BOUND THAN THE ONE OBTAINED IN a))