# Midterm 1, 2021.10.29.
## Probability Theory 1

1./

$B :=$ at least one letter B is found
$M :=$ at least one letter M is found
$E :=$ at least one letter E is found

$\Rightarrow$ We ~~found~~ find at least one of each: $B \cap M \cap E$ 3p

$$\mathbb{P}(B\cap M\cap E) = 1 - \mathbb{P}((B\cap M\cap E)^c) = 1 - \underbrace{\mathbb{P}(B^c \cup M^c \cup E^c)}$$ 2p

We can calculate it using the inclusion-exclusion formula

$$\mathbb{P}(B^c\cup M^c \cup E^c) =$$ 2p

$$= \underbrace{\mathbb{P}(B^c)}_{\left(\frac{25}{26}\right)^{30}} + \underbrace{\mathbb{P}(M^c)}_{\left(\frac{25}{26}\right)^{30}} + \underbrace{\mathbb{P}(E^c)}_{\left(\frac{25}{26}\right)^{30}} - \underbrace{\mathbb{P}(B^c\cap M^c)}_{\left(\frac{24}{26}\right)^{30}} - \underbrace{\mathbb{P}(B^c\cap E^c)}_{\left(\frac{24}{26}\right)^{30}} - \underbrace{\mathbb{P}(M^c\cap E^c)}_{\left(\frac{24}{26}\right)^{30}} + \underbrace{\mathbb{P}(B^c\cap M^c\cap E^c)}_{\left(\frac{23}{26}\right)^{30}} \approx$$ 3p

$\approx 0,68$ $\Rightarrow$ Our probability of winning a T-shirt is $\mathbb{P}(B\cap M\cap E) \approx 0,32$

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2./ $R :=$ we roll 6 with the red die $\qquad \mathbb{P}(R) = 1/6$ (fair die)

$B :=$ we roll 6 with the blue die $\qquad \mathbb{P}(B) = p$ (biased die)

$$p_A := \mathbb{P}(\text{Alice wins}) = \underbrace{\mathbb{P}(R\cap B^c)}_{\text{she won in the first round}} + \underbrace{\mathbb{P}((R\cap B)\cup(R^c\cap B^c))}_{\text{first round was a tie}} \underbrace{\mathbb{P}(R\cap B^c)}_{\text{she won in the second round}} + \ldots =$$

$$= \sum_{k=0}^{\infty} \underbrace{\mathbb{P}((R\cap B)\cup(R^c\cap B^c))^k}_{k\text{-many ties,}} \underbrace{\mathbb{P}(R\cap B^c)}_{\text{she won in the end}} = \sum_{k=0}^{\infty}\left(\frac{1}{6}p + \frac{5}{6}(1-p)\right)^k \frac{1}{6}(1-p) = \frac{1/6\,(1-p)}{1-\left(\frac{p}{6}+\frac{(1-p)5}{6}\right)}$$

Similarly, $p_B := \mathbb{P}(\text{Bella wins}) = \sum_{k=0}^{\infty}\left(\frac{1}{6}p + \frac{5}{6}(1-p)\right)^k \frac{5}{6}p = \frac{5/6\,p}{1-\left(\frac{p}{6}+\frac{(1-p)5}{6}\right)}$ 3p

a/ $p_B = 2p_A \Rightarrow \frac{5}{6}p = \frac{2}{6}(1-p) = \frac{2}{6} - \frac{2}{6}p \Rightarrow \boxed{p = \frac{2}{7}}$ 1p

$$\Rightarrow p_A = \frac{5/42}{1-\left(\frac{2}{42}+\frac{25}{42}\right)} = \frac{5/42}{15/42} = \frac{5}{15} = \frac{1}{3}, \quad p_B = \frac{2}{3}$$ 2p

b/ $X :=$ total number of throws (with both dice together)

$$\mathbb{P}(X=k) = \underbrace{\left(\frac{1}{6}p + \frac{5}{6}(1-p)\right)^{k-1}}_{(k-1)\text{-many ties,}} \underbrace{\left(\frac{1}{6}(1-p)+\frac{5}{6}p\right)}_{\text{then one of them wins}} = \left(\frac{27}{42}\right)^{k-1}\frac{15}{42} \Rightarrow X \sim \text{Geo}\left(\frac{15}{42}\right) \text{ (optimistic)}, \quad \boxed{E[X] = \frac{42}{15}} = \frac{14}{5}$$

2p 2p

Bonus / 5 out of 90 lottery draw

$X$ := the smallest number drawn

$$\mathbb{P}(X=k) = \binom{90-k}{4} \Big/ \binom{90}{5}$$

$1 \leq X \leq 86$

just choose 5 numbers, without repetition or ordering

if $k$ is the smallest, then we must choose the other 4 numbers from the set $\{k+1, \dots, 90\}$
$\Rightarrow$ $90-k$ options for them

$$\Rightarrow \underline{E[X]} = \sum_{k=1}^{86} k\,\mathbb{P}(X=k) =$$

$$= \sum_{k=1}^{86} k \binom{90-k}{4} \Big/ \binom{90}{5} = \underline{\frac{1}{\binom{90}{5}} \sum_{k=1}^{86} \binom{k}{1}\binom{90-k}{4}}$$

3p

To obtain a closed form for $E[X]$ we use the following ~~identity~~ lemma.

lemma: Let $N > M \geq 0$, then

$$\sum_{k=1}^{N-M+1} \binom{k}{1}\binom{N-k}{M-1} = \binom{N+1}{M+1}$$

$N=90$, $M=5$ gives

$$\Rightarrow \sum_{k=1}^{86} \binom{k}{1}\binom{90-k}{4} = \binom{91}{6}$$

That is $\underline{E[X]} = \binom{91}{6} \Big/ \binom{90}{5} = \frac{91!}{6!\,\cancel{85!}} \cdot \frac{5!\,\cancel{85!}}{90!} = \underline{\frac{91}{6}}$ 1p

proof of lemma: (Short combinatorial argument)

$\binom{N+1}{M+1}$ = number of different groups of $M+1$ people, chosen from a population of $N+1$

We ask everyone to queue up, and assign indices for them from 1 to 90. Then we choose someone, say $(k+1)$, and make him assemble the group by choosing $\underbrace{\text{1 person from his left}}_{\binom{k}{1} \text{ choices}}$ and $\underbrace{M-1 \text{ people from his right}}_{\binom{N-k}{M-1} \text{ choices}}$.

Summing up for all possible $k$'s will give us the total number of groups possible.

$$\Rightarrow \sum_{k=1}^{N-M+1} \binom{k}{1}\binom{N-k}{M-1} = \binom{N+1}{M+1}$$

Note that we counted all the possibilities, and each of them only once.

for a given group, we can say that the person with the second lowest index assembled it.

For $1 \leq k \leq N-M+1$ and $i < k$ there is exactly one group that includes both of them.

$\square$