1st make-up midterm 2021. 12. 10.

1. a) French card deck of 52 cards. We draw 8.

$A := \{\text{no diamonds drawn}\}$ $\quad \mathbb{P}(A) = \binom{39}{8} \Big/ \binom{52}{8} =$ 3p

"good cases": choose 8 from the 39 non-diamond cards

"all cases": choose 8 from the whole deck

b) $A_1 := \{\text{at least one spades was drawn}\}$

$A_2 := \{\text{at least one hearths was drawn}\}$

$A_3 := \{\text{at least one diamonds was drawn}\}$ 2p

$A_4 := \{\text{at least one clubs was drawn}\}$

$\{\text{We draw at least 1 from each suit}\}$

$\overset{4}{=} A_1 \cap A_2 \cap A_3 \cap A_4$

$$\mathbb{P}\Big(\bigcap_{i=1}^{4} A_i\Big) = 1 - \mathbb{P}\Big(\Big(\bigcap_{i=1}^{4} A_i\Big)^c\Big) \underset{\text{de Morgan}}{=} 1 - \mathbb{P}\Big(\bigcup_{i=1}^{4} A_i^c\Big) =$$ 2p

(inclusion-exclusion formula

$$= 1 - \underbrace{\mathbb{P}(A_1^c)}_{=\binom{39}{8}/\binom{52}{8}} - \underbrace{\mathbb{P}(A_2^c)}_{=\binom{39}{8}/\binom{52}{8}} - \underbrace{\mathbb{P}(A_3^c)}_{=\binom{39}{8}/\binom{52}{8}} - \underbrace{\mathbb{P}(A_4^c)}_{=\binom{39}{8}/\binom{52}{8}} + \underbrace{\mathbb{P}(A_1^c \cap A_2^c)}_{=\binom{26}{8}/\binom{52}{8}} + \underbrace{\mathbb{P}(A_1^c \cap A_3^c)}_{=\binom{26}{8}/\binom{52}{8}} + \underbrace{\mathbb{P}(A_1^c \cap A_4^c)}_{=\binom{26}{8}/\binom{52}{8}} +$$

$$+ \underbrace{\mathbb{P}(A_2^c \cap A_3^c)}_{=\binom{26}{8}/\binom{52}{8}} + \underbrace{\mathbb{P}(A_2^c \cap A_4^c)}_{=\binom{26}{8}/\binom{52}{8}} + \underbrace{\mathbb{P}(A_3^c \cap A_4^c)}_{=\binom{26}{8}/\binom{52}{8}} - \underbrace{\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c)}_{=\binom{13}{8}/\binom{52}{8}} - \underbrace{\mathbb{P}(A_1^c \cap A_2^c \cap A_4^c)}_{=\binom{13}{8}/\binom{52}{8}} -$$

$$- \underbrace{\mathbb{P}(A_1^c \cap A_3^c \cap A_4^c)}_{=\binom{13}{8}/\binom{52}{8}} - \underbrace{\mathbb{P}(A_2^c \cap A_3^c \cap A_4^c)}_{=\binom{13}{8}/\binom{52}{8}} + \underbrace{\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c)}_{=0} =$$

$$\underset{\text{in a more compact way}}{=} 1 - \binom{4}{1}\binom{39}{8}\Big/\binom{52}{8} + \binom{4}{2}\binom{26}{8}\Big/\binom{52}{8} - \binom{4}{3}\binom{13}{8}\Big/\binom{52}{8} + 0 \approx$$ 3p

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2. $R := \{\text{Richard baked the cake}\}$ $\quad \mathbb{P}(R) = 1/4$

$N := \{\text{Nicole baked the cake}\}$ $\quad \mathbb{P}(N) = 3/4$

$X_t := \{$number of raisins in t slices of cakes (baked by the same person)

The number of raisins is pretty small compared to the mass of the whole slice $\Rightarrow$ X_t has Poisson distribution with parameter λt, where λ depends on who made the sponge cake.
(It is in fact a Poisson process)

a) $\mathbb{P}(\text{no raisins in 4 slices}) = \underline{\mathbb{P}(X_4 = 0)} \underset{\uparrow}{=} \mathbb{P}(X_4=0 \mid R)\,\mathbb{P}(R) + \mathbb{P}(X_4=0 \mid N)\,\mathbb{P}(N) =$

law of total probability 2p

$= e^{-4\cdot 5}\cdot\frac{1}{4} + e^{-4\cdot 3}\cdot\frac{3}{4} \approx \underline{0{,}46\cdot 10^{-5}}$ 1p

$X_4 \sim \mathrm{Poi}(4\cdot 5)$, if Richard bakes the cake, as $\mathbb{E}[X_4] = 4\cdot 5$ in this case 2p

$X_4 \sim \mathrm{Poi}(4\cdot 3)$ if Nicole bakes the cake, as $\mathbb{E}[X_4] = 4\cdot 3$ then.

b) $\underline{\mathbb{P}(N \mid X_1 = 0)} \underset{\uparrow}{=} \dfrac{\mathbb{P}(X_1 = 0 \mid N)\,\mathbb{P}(N)}{\mathbb{P}(X_1 = 0)} \underset{\uparrow}{=} \dfrac{e^{-3}\cdot\frac{3}{4}}{e^{-5}\cdot\frac{1}{4} + e^{-3}\cdot\frac{3}{4}} \approx \underline{0{,}96}$

Bayes-formula 3p

part a)

2p

Bonus) Number of years we wait until the next online semester $\sim \mathrm{Geo}(\frac{1}{3})$

$\Rightarrow$ Number of years we wait until the 5th occurence $\sim \mathrm{Geo}(\frac{1}{3}) + \mathrm{Geo}(\frac{1}{3}) + \mathrm{Geo}(\frac{1}{3}) + \mathrm{Geo}(\frac{1}{3})$

it happens independently of previous semesters

2p

$\Rightarrow$ It has negative binomial distribution with parameters 4 and $\frac{1}{3}$.

That is

$X :=$ number of years we wait until the next occurence

$X \sim \mathrm{Neg}(4, \frac{1}{3})$ by definition $\Rightarrow$ $\underline{\mathbb{E}[X] = 12}$

OR

2p

$X = \sum_{i=1}^{4} X_i$, $X_i \sim \mathrm{Geo}(\frac{1}{3})$ independent random variables

$\underline{\mathbb{E}[X]} = \sum_{i=1}^{4} \mathbb{E}[X_i] = \sum_{i=1}^{4} \frac{1}{1/3} = \underline{12}$