PROBABILITY THEORY 1
MIDTERM 1

1/ $X_1, X_2, X_3, X_4, X_5 \sim \mathrm{Uni}\{1,2,\dots,9\}$

they take each digit with probability $\frac{1}{9}$

$Y = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5$

a/ $$\mathbb{P}(Y \text{ is odd}) = \mathbb{P}(X_1 \text{ is odd}, X_2 \text{ is odd}, \dots, X_5 \text{ is odd}) \overset{\text{they're independent}}{=} \prod_{i=1}^{5} \mathbb{P}(X_i \text{ is odd}) =$$

$$= \prod_{i=1}^{5} \frac{5}{9} = \left(\frac{5}{9}\right)^5 = \frac{3125}{59049} \approx \underline{0{,}0529}$$

the usual $\frac{\#\text{good}}{\#\text{all}}$ formula

b/ $\mathbb{P}(\{70 | Y\}) = \mathbb{P}(\{2|Y\} \cap \{5|Y\} \cap \{7|Y\})$ as "divisible by 70" is equivalent to "divisible by 2 and 5 and 7".

here it means divisibility and not condition

$A := \{70|Y\}, \; A_2 := \{2|Y\}, \; A_5 := \{5|Y\}, \; A_7 := \{7|Y\}$

$$\underline{\mathbb{P}(A)} = \mathbb{P}(A_2 \cap A_5 \cap A_7) = 1 - \mathbb{P}((A_2 \cap A_5 \cap A_7)^c) = 1 - \mathbb{P}(A_2^c \cup A_5^c \cup A_7^c) =$$

$$= 1 - \mathbb{P}(A_2^c) - \mathbb{P}(A_5^c) - \mathbb{P}(A_7^c) + \mathbb{P}(A_2^c \cap A_5^c) + \mathbb{P}(A_2^c \cap A_7^c) + \mathbb{P}(A_5^c \cap A_7^c) - \mathbb{P}(A_2^c \cap A_5^c \cap A_7^c) =$$

inclusion-exclusion formula

$$= 1 - \left(\frac{5}{9}\right)^5 - \left(\frac{8}{9}\right)^5 - \left(\frac{8}{9}\right)^5 + \left(\frac{4}{9}\right)^5 + \left(\frac{4}{9}\right)^5 + \left(\frac{7}{9}\right)^5 - \left(\frac{3}{9}\right)^5 \approx \underline{0{,}1524}$$

just like in part a/

2/ $N_r(t)$:= number of rancid pieces in t kg of walnuts

$N_m(t)$:= number of maggoty pieces in t kg of walnuts

In a kg of walnuts there are a lot of individual pieces, and each of these pieces are rancid/maggoty with a small probability.

$\Rightarrow$ We can model $N_r(t)$ and $N_m(t)$ with Poisson processes.

$\mathbb{E}[N_m(3)] = 1 \Rightarrow 1 = 3 \cdot \lambda_m$, $\underline{\lambda_m = 1/3}$, $\quad \mathbb{P}(N_r(4) = 0) = 0{,}135335$

$N_m(3) \sim \mathrm{Poi}(3 \cdot \lambda_m)$

$N_r(4) \sim \mathrm{Poi}(4 \cdot \lambda_r) \Rightarrow 0{,}135335 = e^{-\lambda_r \cdot 4}$

$\underline{\lambda_r = -\frac{1}{4} \ln 0{,}135335}$

a/ $$\underline{\mathbb{P}(N_r(8) \geq 2)} = 1 - \mathbb{P}(N_r(8) = 0) - \mathbb{P}(N_r(8) = 1) =$$

$$= 1 - e^{-8\lambda_r} - e^{-8\lambda_r} \cdot 8\lambda_r \approx \underline{0{,}97}$$

b) $N(t)$:= number of rancid or maggoty pieces in t kg of walnuts

As $N(t) = N_r(t) + N_m(t)$ we know that $N(t)$ is also a Poisson process with parameter $\lambda_r + \lambda_m$. (Exercise 6.6.)

$$\mathbb{P}(N(t)=0) \geq 0{,}7 \Rightarrow e^{-(\lambda_r+\lambda_m)t} \geq 0{,}7 \Rightarrow t \leq \frac{1}{\lambda_r+\lambda_m} \cdot \ln\frac{10}{7}$$

$N(t) \sim Poi((\lambda_r+\lambda_m)t)$

$t \leq 1{,}2$ kg

BONUS) X := number of records

If the game favours the casino, then $\mathbb{E}[\underbrace{10X - 1000}_{\text{our net winnings}}] < 0$

$\Rightarrow$ By the linearity of the expected value we have $\mathbb{E}X < 100$

To calculate $\mathbb{E}X$ we define the following random variables

$$Y_n := \begin{cases} 1, & a_n \text{ is a record} \\ 0, & \text{otherwise} \end{cases} \qquad \mathbb{P}(Y_n = 1) = \frac{(n-1)!}{n!}$$

$(n-1)!$ ← the arrangements where the biggest value is a_n

$n!$ ← all arrangement of $a_1, \dots, a_n$

$$\mathbb{E}X = \mathbb{E}Y_1 + \mathbb{E}Y_2 + \dots + \mathbb{E}Y_N = \sum_{n=1}^{N} \frac{1}{n} \approx \ln N$$

$$\Rightarrow \ln N < 100 \iff \boxed{N < e^{100}}$$

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