Name: ..... NEPTUN code: ..... Major: .....

Probability Theory 1 exam, January 7th, 2025

Working hours: 100 min. Non-programable calculator without internet connection can be used.
Maximal amount of points (with the Bonus): 110 points, but 100 points are already considered as 100%.

The. 1. (a) *(3 points)* Define when the events A and B are independent.
(b) *(4 points)* Prove that if the events A and B are independent then A and B^c are independent too (where B^c denotes the complement of B).
(c) *(3 points)* Define when the events A and B are conditionally independent with respect to the event C.
(d) *(6 points)* Prove that if the events A and B are conditionally independent with respect to the event C then $\mathbb{P}(A|B \cap C) = \mathbb{P}(A|C)$.

The. 2. *(8+8 points)* Describe Bertrand's paradox! That is, show at least two different methods for choosing a "uniformly distributed" random chord of a circle, and calculate in these cases the probability that the length of the chord is longer than the side-length of the equilateral triangle inscribed in the circle.
Bonus: *(3 points)* If you write down all three different methods learnt in class.

The. 3. (a) *(2+2 points)* Define the covariance and the correlation of two random variables X and Y.
(b) *(1+4 points)* Define the covariance matrix and show that it is positive semi-definite.
(c) *(4 points)* State and prove the Cauchy-Schwartz inequality.
(d) *(5 points)* Show that if the correlation of X and Y is 1 then $Y = a \cdot X + b$ for some $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$.

Prac. 1. *(8+8 points)* We have two urns with red and blue balls. In he first urn, there are 4 red and 5 blue balls, while in the second urn, there are 3 red and 7 blue balls.
(a) Scenario 1: we take a ball from the first urn and put it into the second urn. We then take a ball from the second urn and put it into the first urn. Then, we take another ball from the first urn. What is the probability that this ball is red?
(b) Scenario 2: we take 1-1 ball from the two urns at the same time, and switch them. Then we take a ball from the first urn. What is the probability that this ball is red?

Prac. 2. The ambulance and fire departments are not assigned to the same areas. Kukutyin (800 inhabitants) and Piripócs (1200 inhabitants) are part of the ambulance rescue area "KP", while Szecső (900 inhabitants) and Pajkaszeg (1300 inhabitants) are part of the ambulance rescue area "SP". Kukutyin and Szecső are in the fire alarm area "KS", and Piripócs and Pajkaszeg are in the fire alarm area "PP". On New Year's Eve, each resident, independent of the rest of the population, has a 0.001 chance of being a victim of a fireworks accident. The ambulance and the fire-fighters are both alarmed in case of each fireworks accident. *Instructions:* Use the Poisson approximation to solve the following problems.
(a) *(5 points)* What is the probability that the fire department in the fire alarm area "PP" will receive at least two alarms?
(b) *(6 points)* Assuming that the ambulance in the ambulance rescue area "KP" received exactly three alarms, what is the probability that there were exactly two firework accidents in Piripócs?
(c) *(6 points)* Denote by X the number of alarms received by the fire department in the fire alarm area "PP", and denote Y the number of alarms received by the ambulance in the rescue area "KP". $\mathrm{Cov}(X, Y) = ?$
Bonus: *(4+3 points)* Calculate $\mathbb{E}(X|Y)$ and $\mathbb{D}^2(X|Y)$.

Prac. 3. *(17 points)* Two groups of researchers want to measure the charge of a particle. The first group takes 5 measurements and averages it, the second group takes 7 measurements and also averages it. The measurements are independent and identically distributed with normal distribution, the expected value is the actual charge μ of the particle, and the standard deviation is σ. What is the probability that the first group's result is more accurate than the second group's result?

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000