Name: .... NEPTUN code: .... Major: ....

Lecturer: Balázs Bárány

Probability Theory 1 exam, December 10th, 2024

Working hours: 100 min. Non-programable calculator without internet connection can be used.
Maximal amount of points (with the Bonus): 110 points, but 100 points are already considered as 100%.

The. 1. (a) *(3 points)* Define the conditional probability of the event A conditioned on event B.

(b) *(5 points)* Let $E_1, \dots, E_n$ be arbitrary events. Give and prove the relation between the probability $\mathbb{P}(\cap_{i=1}^n E_i)$ and the probabilities $\mathbb{P}(E_1), \mathbb{P}(E_2 \mid E_1), \dots, \mathbb{P}(E_n \mid E_1 \cap \dots \cap E_{n-1})$!

(c) *(3 points)* When do we call a countable collection of events partition (complete event system)?

(d) *(5 points)* State and prove the theorem of complete event systems (also known as *law of total probability*)!

The. 2. (a) *(5 points)* State and prove Markov's inequality.

(b) *(5 points)* State and prove Chebisev's inequality.

(c) *(7 points)* State and prove the Weak Law of Large Numbers.

The. 3. (a) *(3 points)* Define the Poisson distribution,

(b) *(3 points)* Prove that it is indeed a distribution, that is, if X has Poisson distribution then $\sum_{k=0}^{\infty} \mathbb{P}(X = k) = 1$,

(c) *(3 points)* Compute the moment generating function of the Poisson distribution,

(d) *(4 points)* Compute the expected value of the random variable X with Poisson distribution,

(e) *(4 points)* Compute the variance of X.

Prac. 1. The Advent calendar is a 24-day calendar from the 1st to the 24th of December in which a small candy is hidden for each day: kids open one door of their calendar each morning. The little angels put four types of chocolate in our Advent calendar: white, milk, dark and nougat, six of each.

(a) *(8 points)* Find the probability that until December 10th (included) we have found at least one of all four types of chocolate.

(b) *(9 points)* Determine the covariance of the number of dark and nougat chocolates we found until 10th of December (included).

Bonus: *(10 points)* The Santa Claus got a little bit dizzy and he visited us not on Christmas but on a random day uniformly chosen between 1st and 24th of December. Find the expected value and the variance of the number of milk chocolates we found until the day of Santa Claus' visit. *Hint 1:* If we consider a random variable X with hypergeometric distribution, where N is the population size, K is the number of success states in the population, n is the sample size and X is the number of success states in the sample then $\mathrm{Var}(X) = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}$. *Hint 2:* If $a < b$ are positive integers and Y is uniformly distributed on the set $\{a, a+1, \dots, b-1, b\}$ then $\mathrm{Var}(Y) = \frac{(b-a+1)^2-1}{12}$.

Prac. 2. *(16 points)* Choose a point uniformly and randomly in the interior of an equilateral triangle of unit side length. Denote by D the distance of the point from the closest side of the triangle. Determine the distribution function F and the density function f of D. Plot the graph of the distribution function $F : \mathbb{R} \to [0, 1]$.

Prac. 3. On the exam of a subject, the expected value of a student's score is 74 points, while the standard deviation is 14. Assume that the scores of students are independent of each other. The first exam was taken by 36 students and the second was taken by 49. Estimate the following probabilities:

(a) *(9 points)* the average score on the two exams differ by more than 2 points,

(b) *(8 points)* the average score of the second exam is closer to 74 than that of the first exam's average score.

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000