Name: .... NEPTUN code: .... Major: ....

Probability Theory 1 exam, December 17th, 2024

Working hours: 100 min. Non-programable calculator without internet connection can be used.
Maximal amount of points (with the Bonus): 110 points, but 100 points are already considered as 100%.

The. 1.
(a) *(3 points)* State the axioms of a probability measure.
(b) *(4 points)* State the inclusion-exclusion (sieve) formula for n events.
(c) *(3 points)* Prove the inclusion-exclusion (sieve) formula for 2 events.
(d) *(6 points)* The 5th class of a primary school with 29 students is preparing for Christmas. They decided that they will give a small gift for each other. To decide who gives a gift to whom, they write their names on a little slip of a paper and they put it in a hat. They shuffle the slips in the hat and everybody draws one name randomly. What is the probability that no student will draw his/her own name? *Instruction:* The final result is not enough, present the computation!

The. 2. Let $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ with $\sigma \neq 0$.

(a) *(3 points)* Write down the probability density function (p.d.f.) of the $N(\mu, \sigma^2)$ distribution.
(b) *(7 points)* Prove that the p.d.f. of the standard normal distribution $N(0, 1)$ is indeed a p.d.f.
(c) *(6 points)* Prove that if Y has distribution $N(0, 1)$ then $X = \sigma \cdot Y + \mu$ has distribution $N(\mu, \sigma^2)$.

The. 3.
(a) *(2+2 points)* Define the covariance and the correlation of two random variables X and Y.
(b) *(2 points)* Define the covariance matrix of the random variables $X_1, \ldots, X_n$.
(c) *(5 points)* Show that the covariance matrix is positive semi-definite.
(d) *(6 points)* State and prove the Cauchy-Schwartz inequality.

Bonus: *(10 points)* Let X be a random variable uniformly distributed on the interval $[6, 9]$. The distribution of the random variable Y is unknown but we know that its expected value is 10, its deviation is $\sqrt{3}$ and $\mathrm{Cov}(X, Y) = 3/2$. Using these informations, what can we say about the probability $\mathbb{P}(Y \geq 9)$?

Prac. 1. *(8+8 points)* We put five butterfly traps on a field far from each other. Every trap, independently of the others, opens with probability $2/3$. If a trap does not open it won't catch any butterfly. If a trap is open then every butterfly nearby flies into the trap with a small probability independently of the other butterflies. For an opened trap, the expected time between trapping two butterflies is 20 minutes. The traps are on the field for one hour and then we collect them. What is the expected value and variance of the caught butterflies?

Prac. 2. *(17 points)* Let the side-lengths of a rectangle be 4 and 3. Let us choose two points, one-one on both sides with length 4, uniformly, randomly and independently. Denote by X the distance of these points. Find the distribution function and the density function of X. Plot the graph of the distribution function.

Prac. 3. Santa Claus drives his reindeer-powered flying sleigh, while baby Jesus drops gifts from the back seat through the chimneys of family homes. Every time he targets the middle of the chimney-holes (of which we assume that it is the origin of a planar coordinate system), but the point where the gift lands is a two-dimensional jointly normal random variable with expected value $(0, 0)$ and covariance matrix 3-times the identity matrix. What is the probability that he hits the chimney-hole if

(a) *(8 points)* the chimney-hole is square shaped with vertices $(0, 1), (1, 0), (0, -1)$ and $(-1, 0)$?
(b) *(9 points)* the chimney-hole is disc shaped centered at the origin and with radius $3/2$?

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000