$\boxed{\text{THE } 1}$ a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

SINCE A AND B ARE INDEP.

b) WANT: $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) \cdot \mathbb{P}(B^c)$

PROOF: $\mathbb{P}(A) \cdot \mathbb{P}(B^c) = \mathbb{P}(A) \cdot (1 - \mathbb{P}(B)) = \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B) =$

$= \mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A \setminus B) = \mathbb{P}(A \cap B^c)$

c) IF $\mathbb{P}(A \cap B | C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

d) WANT: $\dfrac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} \overset{\text{WANT}}{=} \dfrac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)}$

PROOF: WE KNOW: $\dfrac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \dfrac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \cdot \dfrac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$

THUS: $\mathbb{P}(A \cap B \cap C) = \dfrac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \cdot \mathbb{P}(B \cap C)$

REARRANGING THIS WE OBTAIN WHAT WE WANTED TO PROVE.

THE 2

(A): CHOOSE THE ENDPOINTS OF THE CHORD INDEPENDENTLY FROM THE PERIMETER OF THE CIRCLE. BY ROTATION INVARIANCE, WE MIGHT ASSUME THAT THE FIRST POINT IS THIS

THUS THE OTHER POINT MUST FALL IN THE RED ARC

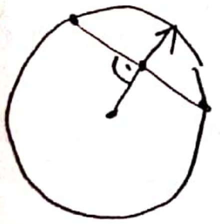
THUS $\boxed{p = \frac{1}{3}}$

(B): CHOOSE THE MIDPOINT OF THE CHORD UNIFORMLY FROM THE DISC. SINCE THE DISTANCE OF THE MIDPOINT OF THE TRIANGE AND THE MIDPOINT OF THE CIRCLE IS $\frac{\text{RADIUS}}{2}$,

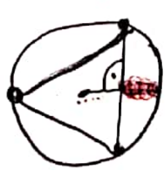
THUS THE MIDPOINT OF THE CHORD MUST FALL IN

THE RED DISC, THUS $\boxed{p = \frac{1}{4}}$

(C): CHOOSE A DIRECTION UNIFORMLY FROM $[0, 2\pi]$. CHOOSE A CHORD THAT IS PERPENDICULAR TO THIS DIRECTION BY CHOOSING THE INTERSECTION POINT UNIFORMLY.



ROTATIONAL INVARIANCE:



$\boxed{p = \frac{1}{2}}$

THE 3 a) $\operatorname{Cov}(X, Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$

$$\operatorname{Corr}(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)} \cdot \sqrt{\operatorname{Var}(Y)}}$$

---

b) $\underline{\underline{C}} = (c_{i,j})_{i,j=1}^{n}$ $\quad c_{i,j} = \operatorname{Cov}(X_i, X_j)$ $\quad$ BI-LINEARITY

WANT: $\underline{v}^T \cdot \underline{\underline{C}} \cdot \underline{v} \geq 0 \quad \forall \underline{v} \in \mathbb{R}^n, \quad \underline{v} = (v_1, \dots, v_n)$

PROOF: $\underline{v}^T \cdot \underline{\underline{C}} \cdot \underline{v} = \sum_{i,j=1}^{n} c_{i,j} \cdot v_i \cdot v_j = \sum_{i,j=1}^{n} \operatorname{Cov}(v_i \cdot X_i, v_j \cdot X_j) =$

$$= \operatorname{Cov}\left(\sum_{i=1}^{n} v_i \cdot X_i, \sum_{j=1}^{n} v_j \cdot X_j\right) = \operatorname{Var}\left(\sum_{i=1}^{n} v_i \cdot X_i\right) \geq 0$$

---

c) $\underline{\underline{C}} = \begin{pmatrix} \operatorname{Var}(X) & \operatorname{Cov}(X, Y) \\ \operatorname{Cov}(X, Y) & \operatorname{Var}(Y) \end{pmatrix} \leftarrow$ POSITIVE SEMI-DEFINITE

THUS $\det(\underline{\underline{C}}) \geq 0$, I.E.: $\operatorname{Var}(X) \cdot \operatorname{Var}(Y) - \operatorname{Cov}(X, Y)^2 \geq 0$

THUS $\boxed{|\operatorname{Cov}(X, Y)| \leq \sqrt{\operatorname{Var}(X)} \cdot \sqrt{\operatorname{Var}(Y)}} \leftarrow$ CAUCHY-SCHWARTZ

---

d) LET $a = \sqrt{\frac{\operatorname{Var}(Y)}{\operatorname{Var}(X)}}$ AND $b = \mathbb{E}(Y) - a \cdot \mathbb{E}(X)$

LET $\boxed{Z_1 := Y - (a \cdot X + b)}$ THEN $\boxed{\mathbb{E}(Z_1) = 0}$ AND

$$\operatorname{Var}(Z_1) = \operatorname{Var}(Y - a \cdot X) = \operatorname{Var}(Y) - 2a \cdot \underbrace{\operatorname{Cov}(X, Y)}_{= \sqrt{\operatorname{Var}(X)} \cdot \sqrt{\operatorname{Var}(Y)}} +$$

$$a^2 \cdot \operatorname{Var}(X) = \operatorname{Var}(Y) - 2 \cdot \operatorname{Var}(Y) + \operatorname{Var}(Y) = 0$$

SINCE CORR = 1

THUS $\boxed{Z_1 = 0}$ ✓

PRAC 1

a) R_i = THE i'TH BALL THAT WE PICK IS RED

$\mathbb{P}(R_3) =$

$$\underbrace{\mathbb{P}(R_1 \cap R_2 \cap R_3)}_{\frac{4}{9}\cdot\frac{4}{11}\cdot\frac{4}{9}} + \underbrace{\mathbb{P}(R_1^c \cap R_2 \cap R_3)}_{\frac{5}{9}\cdot\frac{3}{11}\cdot\frac{5}{9}} + \underbrace{\mathbb{P}(R_1 \cap R_2^c \cap R_3)}_{\frac{4}{9}\cdot\frac{7}{11}\cdot\frac{3}{9}} + \underbrace{\mathbb{P}(R_1^c \cap R_2^c \cap R_3)}_{\frac{5}{9}\cdot\frac{8}{11}\cdot\frac{4}{9}}$$

$$= \frac{383}{891} = 0.429854\ldots$$

b) $\mathbb{P}(R_3) =$

$$\underbrace{\mathbb{P}(R_1 \cap R_2 \cap R_3)}_{\frac{4}{9}\cdot\frac{3}{10}\cdot\frac{4}{9}} + \underbrace{\mathbb{P}(R_1^c \cap R_2 \cap R_3)}_{\frac{5}{9}\cdot\frac{3}{10}\cdot\frac{5}{9}} + \underbrace{\mathbb{P}(R_1 \cap R_2^c \cap R_3)}_{\frac{4}{9}\cdot\frac{7}{10}\cdot\frac{3}{9}} + \underbrace{\mathbb{P}(R_1^c \cap R_2^c \cap R_3)}_{\frac{5}{9}\cdot\frac{7}{10}\cdot\frac{4}{9}}$$

$$= \frac{347}{810} = 0.428395\ldots$$

GYAK 1

$$A_i = \left\{ \begin{array}{c} \text{ANDRÁSNAK AZ } i\text{-EDIK FIGURÁBÓL} \\ \text{MIND A NÉGY SZÍN A KEZÉBEN LESZ} \end{array} \right\}$$

$$\mathbb{P}\left(\bigcap_{i=1}^{8} A_i^c \right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{8} A_i \right) = ☺$$

SZITA

$$\mathbb{P}\left(\bigcup_{i=1}^{8} A_i \right) = \sum_{\substack{I \subseteq [8] \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot \mathbb{P}\left(\bigcap_{i \in I} A_i \right) = ☹$$

$$\mathbb{P}\left(\bigcap_{i \in I} A_i \right) = \begin{cases} 0, & \text{HA } |I| \geq 3 \\ \dfrac{\binom{24}{2}}{\binom{32}{10}}, & \text{HA } |I| = 2 \\ \dfrac{\binom{28}{6}}{\binom{32}{10}} & \text{HA } |I| = 1 \end{cases}$$

$$☹ = \binom{8}{1} \cdot (-1)^2 \cdot \frac{\binom{28}{6}}{\binom{32}{10}} + \binom{8}{2} \cdot (-1)^3 \cdot \frac{\binom{24}{2}}{\binom{32}{10}}$$

$$☺ = 1 - 8 \cdot \frac{\binom{28}{6}}{\binom{32}{10}} + \binom{8}{2} \cdot \frac{\binom{24}{2}}{\binom{32}{10}} \approx 0.9534$$

PRAC 2

a) POPULATION OF PP IS 2500

$$\mathbb{X} \sim BIN\left(2500, \frac{1}{1000}\right) \approx POI(2.5)$$

$$\mathbb{P}(\mathbb{X} \geq 2) = 1 - \mathbb{P}(\mathbb{X} \leq 1) = 1 - e^{-2.5} - e^{-2.5} \cdot 2.5$$

$$= 0.7127\ldots$$

---

b) $\mathbb{Y} \sim BIN\left(2000, \frac{1}{1000}\right) \approx POI(2)$

$\mathbb{Y}_K$ = NUMBER OF KUKUTYIN ACCIDENTS

$\mathbb{Y}_P$ = —"— —"— PIRIPÓCS —"—

INDEP!

$$\boxed{\mathbb{Y} = \mathbb{Y}_K + \mathbb{Y}_P} \qquad \mathbb{Y}_K \approx POI(0.8), \quad \mathbb{Y}_P \approx POI(1.2)$$

$$\mathbb{P}(\mathbb{Y}_P = 2 \mid \mathbb{Y} = 3) = \frac{\mathbb{P}(\mathbb{Y}_P = 2, \mathbb{Y} = 3)}{\mathbb{P}(\mathbb{Y} = 3)} = \frac{\mathbb{P}(\mathbb{Y}_P = 2) \cdot \mathbb{P}(\mathbb{Y}_K = 1)}{\mathbb{P}(\mathbb{Y} = 3)} =$$

$$= \frac{e^{-1.2} \cdot \frac{(1.2)^2}{2!} \cdot e^{-0.8} \cdot \frac{0.8}{1!}}{e^{-2} \cdot \frac{2^3}{3!}} = \frac{\frac{(1.2)^2}{2} \cdot 0.8}{(8/6)} = 0.432$$

---

c) $\mathbb{X}_P$ = NUMBER OF PAJKASZEG ACCIDENTS $\sim POI(1.3)$

$$\boxed{\mathbb{X} = \mathbb{X}_P + \mathbb{Y}_P} \qquad Cov(\mathbb{X}, \mathbb{Y}) = Cov(\mathbb{X}_P + \mathbb{Y}_P, \mathbb{Y}_K + \mathbb{Y}_P) =$$

BIL.

$$\stackrel{\Downarrow}{=} \underbrace{Cov(\mathbb{X}_P, \mathbb{Y}_K)}_{0} + \underbrace{Cov(\mathbb{X}_P, \mathbb{Y}_P)}_{0} + \underbrace{Cov(\mathbb{Y}_P, \mathbb{Y}_K)}_{0} + Cov(\mathbb{Y}_P, \mathbb{Y}_P) = ☺$$

- ☺ $= Var(\mathcal{Y}_p) = Var(POI(1.2)) = 1.2$
-

<u>BONUS</u>: $\mathbb{E}(\mathcal{X}|\mathcal{Y}) = \underbrace{\mathbb{E}(\mathcal{X}_p|\mathcal{Y})}_{\overset{\boxed{\text{INDEPENDENCE}}}{=} \mathbb{E}(\mathcal{X}_p) = 1.3} + \underbrace{\mathbb{E}(\mathcal{Y}_p|\mathcal{Y})}_{☺}$

IF WE HAVE $\mathcal{Y} \sim POI(2)$ ACCIDENTS AND COLOR THEM INDEPENDENTLY EITHER „KUKUTYIN" (WITH PROB 0.4) OR „PIRIPÓCS" (WITH PROB 0.6) THEN BY THE COLORING PROPERTY OF POISSON, WE GET = $\mathcal{Y}_K \sim POI((0.4)\cdot 2) \sim POI(0.8)$, $\mathcal{Y}_p \sim POI((0.6)\cdot 2) \sim POI(1.2)$

JUST LIKE ON PAGE 5.

THUS IF $\mathcal{Y} = m$ THEN THE CONDITIONAL DISTRIBUTION OF $\mathcal{Y}_p$ IS $BIN(m, 0.6)$,

THUS $\mathbb{E}(\mathcal{Y}_p | \mathcal{Y} = m) = m \cdot 0.6$, THUS ☺ $= (0.6)\cdot \mathcal{Y}$

<u>SIMILARLY</u>: $Var(\mathcal{X}|\mathcal{Y}) =$

$$\underbrace{Var(\mathcal{X}_p|\mathcal{Y})}_{Var(\mathcal{X}_p) = 1.3} + \underbrace{Var(\mathcal{Y}_p|\mathcal{Y})}_{=(0.6)\cdot(1-0.6)\cdot\mathcal{Y}}$$

$\boxed{\text{PRAC 3}}$ $\bar{X}$ = ESTIMATE OF FIRST GROUP

$\bar{Y}$ = — " — — " — SECOND — " —

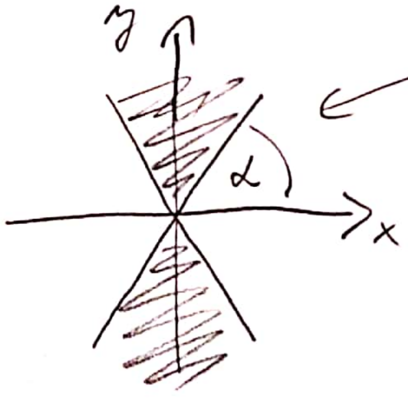
$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{5}\right) \qquad \bar{Y} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{7}\right)$$

$$\bar{X}^* = \frac{\bar{X} - \mu}{\sqrt{\sigma^2/5}} \qquad \bar{Y}^* = \frac{\bar{Y} - \mu}{\sqrt{\sigma^2/7}}$$

THEN $(\bar{X}^*, \bar{Y}^*)$ HAS 2-DIM STANDARD NORMAL DISTR.

$$\mathbb{P}\left(|\bar{X} - \mu| < |\bar{Y} - \mu|\right) = \mathbb{P}\left(\left|\bar{X}^* \cdot \sqrt{\frac{\sigma^2}{5}}\right| < \left|\bar{Y}^* \cdot \sqrt{\frac{\sigma^2}{7}}\right|\right) =$$

$$= \mathbb{P}\left(\left|\frac{\bar{X}^*}{\sqrt{5}}\right| < \left|\frac{\bar{Y}^*}{\sqrt{7}}\right|\right) = \mathbb{P}\left(|\bar{Y}^*| > \sqrt{\frac{7}{5}} \cdot |\bar{X}^*|\right) = ☹$$



SLOPES $= \pm \sqrt{\frac{7}{5}}$

$$\alpha = \arctan\left(\sqrt{\frac{7}{5}}\right) = 0.869$$

$$☹ = \mathbb{P}\left((\bar{X}^*, \bar{Y}^*) \in \text{SHADED}\right) = \frac{2\pi - 4 \cdot \alpha}{2\pi} =$$

$$= 1 - \frac{2\alpha}{\pi} = 0.44669$$