PROB. THEO. 1. | RETAKE 2ND MIDT. | 2024. 12.03.

(1) INTENSITY: $1/2$ MUSHROOM / KM^2

$\mathcal{A}$:= NUMBER OF MUSHROOMS FOUND BY ADAM

$\mathcal{B}$:= —"— —"— —"— —"— BY BRIAN

$\boxed{\mathcal{C} := \mathcal{A} + \mathcal{B}}$ = NUM. OF MUSH. FOUND TOGETHER

$\mathcal{A} \sim POI\left(\frac{5}{2}\right)$, $\mathcal{B} \sim POI\left(\frac{3}{2}\right)$ $\mathcal{C} \sim POI(4)$

$\mathcal{A}$ AND $\mathcal{B}$ ARE INDEP. RANDOM VARIABLES.

a) $\mathbb{P}(\mathcal{C} \geq 3) = 1 - \mathbb{P}(\mathcal{C} \leq 2) =$

$$1 - e^{-4} - e^{-4} \cdot 4 - e^{-4} \cdot \frac{4^2}{2} = 0.762$$

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b) $\mathbb{P}(\mathcal{A} = 2, \mathcal{B} = 1) = \mathbb{P}(\mathcal{A} = 2) \cdot \mathbb{P}(\mathcal{B} = 1) =$

$$= \left(e^{-5/2} \cdot \frac{\left(\frac{5}{2}\right)^2}{2}\right) \cdot \left(e^{-3/2} \cdot \frac{3}{2}\right) = 0.0858$$

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c) $\mathbb{P}(\mathcal{A} > \mathcal{B} \mid \mathcal{C} = 3) =$

$$\mathbb{P}(\{\mathcal{A} = 3, \mathcal{B} = 0\} \cup \{\mathcal{A} = 2, \mathcal{B} = 1\} \mid \mathcal{C} = 3) =$$

$$\frac{\mathbb{P}(\mathcal{A} = 3) \cdot \mathbb{P}(\mathcal{B} = 0) + \mathbb{P}(\mathcal{A} = 2) \cdot \mathbb{P}(\mathcal{B} = 1)}{\mathbb{P}(\mathcal{C} = 3)} = ☆$$

$$\star = \frac{e^{-5/2}\cdot\frac{(5/2)^3}{6}\cdot e^{-3/2} + e^{-5/2}\cdot\frac{(5/2)^2}{2}\cdot e^{-3/2}\cdot\frac{3}{2}}{e^{-4}\cdot\frac{4^3}{6}} =$$

$$= \frac{(\frac{5}{2})^3/6 + (5/2)^2/2\cdot\frac{3}{2}}{4^3/6} = \frac{175}{256} = 0.6836$$

BONUS: THE EVENTS $\{X<Y<Z\}$, $\{X<Z<Y\}$, $\{Y<X<Z\}$, $\{Y<Z<X\}$, $\{Z<X<Y\}$, $\{Z<Y<X\}$ HAVE THE SAME PROBABILITY SINCE X, Y, Z ARE I.I.D. SINCE X, Y, Z ARE ALL DISTINCT WITH PROBABILITY 1, EXACTLY ONE OF THE ABOVE 6 EVENTS WILL OCCUR WITH PROBABILITY 1. THUS ALL OF THE ABOVE EVENTS HAVE PROBABILITY $1/6$.

IN PARTICULAR:

$$\mathbb{P}(\{X<Y\}\cap\{Y<Z\}) = \frac{1}{6}$$

② a) $1 = \int_0^\infty \frac{c}{(1+x)^4} dx = c \cdot \left[-\frac{1}{3} \cdot (1+x)^{-3} \right]_0^\infty = \frac{c}{3}$

THUS $\boxed{C=3}$.

b) IF X IS THE LIFE SPAN OF A LIGHTBULB,

THEN $\mathbb{E}(1+X) = \int_0^\infty (1+x) \cdot f(x)\, dx = \int_0^\infty \frac{3}{(1+x)^3} dx =$

$= 3 \cdot \left[-\frac{1}{2} \cdot (1+x)^{-2} \right]_0^\infty = \frac{3}{2} = \mathbb{E}(1+X),$

THUS $\mathbb{E}(X) = \frac{3}{2} - 1 = \frac{1}{2}$

$\mathbb{E}\left((1+X)^2 \right) = \int_0^\infty (1+x)^2 \cdot f(x)\, dx = \int_0^\infty \frac{3}{(1+x)^2} dx = 3$

$3 = \mathbb{E}\left((1+X)^2\right) = \mathbb{E}\left(1 + 2X + X^2\right) =$

$= 1 + 2 \cdot \mathbb{E}(X) + \mathbb{E}(X^2) = 1 + 2 \cdot \frac{1}{2} + \mathbb{E}(X^2)$

THUS $\mathbb{E}(X^2) = 1$ THUS $\mathrm{Var}(X) = 1 - \left(\frac{1}{2}\right)^2 = 3/4$

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c) $\mathbb{P}(\text{EXCELLENT}) = \int_1^\infty \frac{3}{(1+x)^4} dx = \left[-(1+x)^{-3} \right]_1^\infty = \frac{1}{8}$

$Y :=$ NUMBER OF EXCELLENT LIGHTBULBS OUT OF 100

$Y \sim \mathrm{BIN}\left(100, \frac{1}{8}\right)$ T.B.C.

$$\mathbb{P}(Y \geq 17) = \mathbb{P}\left(\frac{Y - 100\cdot\frac{1}{8}}{\sqrt{100\cdot\frac{1}{8}\cdot\frac{7}{8}}} \geq \frac{17 - 100\cdot\frac{1}{8}}{\sqrt{100\cdot\frac{1}{8}\cdot\frac{7}{8}}}\right) \approx$$

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$$1 - \Phi\left(\frac{17 - 100\cdot\frac{1}{8}}{10\cdot\frac{\sqrt{7}}{8}}\right) =$$

$$= 1 - \Phi(1.36) =$$

$$= 1 - 0.9131 = 0.0869$$