DIPLOMA THESIS

Conformal invariance of critical percolation on the triangular lattice

Balázs Ráth

Supervisor: Bálint Tóth
professor

Institute of Mathematics,
Budapest University of
Technology and Economics

BUTE
2005.

Contents

1 Introduction

Percolation theory is concerned with the connectivity properties of large random graphs: classical percolation processes consist of random subgraphs of finite dimensional lattice graphs spanned by a random vertex set of "density" p. The theory investigates how the probability of the occurence of certain big connected components depends on the parameter p. The mathematics of percolation on planar graphs is developed enough to verify conjectures of statistical phisical flavour.

We can observe the phenomenon of *phase transition*: there is a critical probability p_c such that for $p < p_c$ the resulting random subgraph has connected components of finite size, but when $p > p_c$, it has an infinite component of positive density which penetrates the whole plane (see [4] for the proof of these nontrivial facts). The most interesting phenomena occur when $p = p_c$: this is the case when the probability of certain properties of the random connected subgraphs become scale invariant.

My diploma thesis investigates conformal invariance of crossing probabilities of 2-dimensional critical percolation processes. The fruitful idea of identifying the plane with the set of complex numbers $\mathbb{C}$ is of statistical phisical origin: the heuristics behind conformal invariance is a natural generalization of rotation and scale invariance. John Cardy used these heuristic ideas in [3] to give a formula that determines the exact values of the crossing probabilities between the opposite sides of a conformal rectange filled with a "conformally invariant infinitesimal lattice". Stanislav Smirnov succeeded in generalizing and proving Cardy's conjecture in [8] for a special percolation model: $\frac{1}{2} - \frac{1}{2}$ Bernoulli site percolation on the regular triangular lattice. Smirnov's result was a real breakthrough in the subject of conformal invariance (he received the Clay Research Award in 2001 for this achievement), and the backbone of my diploma thesis consists of a presentation of his methods. Nevertheless it contains some simplifications compared to Smirnov's original paper.

Section 2 provides the reader with the necessary definitions to understand conformal invariance of crossing probabilities and the Cardy formula, but it is important to emphasize that the general form of Cardy's conjecture is only presented in Section 7, because extra care is needed for the definition of a "good embedding" of a general lattice. These definitions are unnecessary for the formulation of Theorem 2.2: in fact, the symmetries of the natural embedding of the regular triangular lattice play an important (and somewhat miraculous) role in the proof of Smirnov's theorem, and the lack of these symmetries is the reason why Cardy's conjecture remains unproved for other critical percolation models.

Section 3 is a concise summary of the topological facts necessary for the investigation of site percolation on triangulated planar graphs. Duality and the extremal path property is usually defined in the more general setting of mosaic graphs, but it is enough for our purposes to present the self-dual case only.

In Section 4, I present Smirnov's theorem of uniform convergence of the functions H^{δ}_{β} to a conformally invariant harmonic limit, but the formulation is different from that of Smirnov: I used the idea of Vincent Beffara (École Normale Supérieure de Lyon) to reveal the fact that an appropriate linear combination of the conformally invariant functions is a confomal mapping itself. I also present the partial differential equation that served as the definition of the harmonic function h_β in Smirnov's original paper.

In Section 5 we show that we can apply the Arzela-Ascoli theorem to prove that the set of functions $\{H_\beta^\delta : \delta > 0\}$ form a precompact subset in the Banach space of continous functions. I altered some technical details of the original statement and proof, because the methods presented by Smirnov are sufficient to prove Hölder-continuity of H_β^δ only if $\partial\Omega$ is a Hölder-continous Jordan curve itself.

Compactness arguments combined with the theory of conformal mappings enable us to determine the limit of the functions H_β^δ. The only thing that remains to be proved is a discrete, approximate version af analiticity: in Section 6 we present Smirnov's proof of a combinatorical identity that can be interpreted as a discrete version of the Caucy-Riemann equations, we introduce Riemann sums to approximate contour integrals and conclude the proof of Smirnov's theorem by showing that the contour integrals almost vanish. The discrete integration method developed in this diploma thesis is simpler and possibly more generally applicable than that of Smirnov. Nevertheless our methods are still insufficient to prove a generalization of Smirnov's theorem: Section 7 discusses the universal form of Cardy's conjecture and the way it defines a complex structure of the plane.

Acknowledgements

I am grateful to my supervisor, professor Bálint Tóth for introducing me to the subject of percolation theory and for helping me during my research on the topic and the writing of this diploma thesis.

I would like to thank Dr. Vincent Beffara for sharing his ideas with me, Ambrus Zsbán for his valuable comments about harmonic conjugate triplets and Anna Rudas for reading the manuscript.

2 Cardy's Conjecture

This section is an itroduction to the concept of conformal invariance: we define critical Bernoulli site percolation on the regular triangular lattice, state the fundamental results of the theory of conformal mappings and present Cardy's conjecture both in its original form and in Carleson's reformulation.

2.1 Crossing probabilities of the rhombus

Let us begin with the simplest example of the theory of critical percolation on triangulated planar graphs. Although this is the only case when elementary methods are sufficient, it has strong suggestive power and can be viewed as the verification of Cardy's conjecture in a very special case.

Let us consider a triangulation of a rhombus with equilateral triangles (see Figure 2.1). This is a planar graph with triangular faces. We colour all the vertices

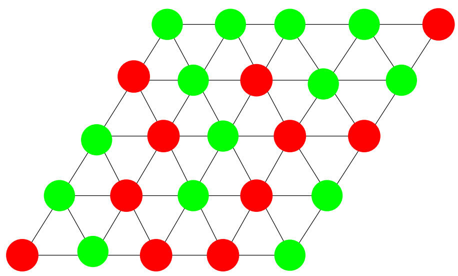

Figure 2.1: A left-right red crossing

of this graph green or red with independent fair coin tosses, and investigate the connectivity properties of the resulting monochromatic subgraphs.

Lemma 2.1. *The probability of the event that a vertex on the top side of the rhombus is connected to a vertex on the bottom side with a green simple path is* $\frac{1}{2}$.

Proof. If there is a green path connecting the top side to the bottom side, then the left side cannot be connected to the right side by a red path. This is just a trivial topological property of planar graphs, but the converse holds only for triangulations: if there is no green path connecting the top side to the bottom side, then there is a red path connecting the left side to the right side. This is called self-duality, and we will discuss it in more detail in Proposition 3.1.

If A is the event that top-bottom green crossing occurs and B is the event that a left-right red crossing occurs, then A and B form a complete system of disjoint events on our probability space, so $\mathbf{P}(A)+\mathbf{P}(B)=1$. But the probability of B is the same as the probability of a left-right green crossing, because reversing the colour of the vertices of the graph won't change the probabilities. Finally, the image of a left-right green crossing under the reflection in the diagonal of the rhombus is a top-bottom green crossing, and this transformation also leaves the probabilities invariant. These facts imply $\mathbf{P}(A) = \mathbf{P}(B)$, so $\mathbf{P}(A) = \frac{1}{2}$. $\square$

This result might be stated in a slightly different form: if the sides of the rhombus are of unit length, and the mesh δ of the regular triangular lattice (the length of the side of an elementary triangle) goes to 0, the limit of crossing probabilities between the opposite sides exists and is invariant under the symmetries of the rhombus. Two natural questions arise:

- Does the limit exist if we replace the rhombus by another plane figure?
- What is the invariance group of the limit of crossing probabilities?

Maybe the most important idea in Lemma 2.1 is that the two questions can only be answered simultaneously.

2.2 Conformal invariance of crossing probabilities

We need some definitions in order to formulate Cardy's conjecture precisely:

The topological generalization of a rhombus is a singly connected domain $\Omega \subseteq \mathbb{C}$, bounded by a Jordan-curve, with four labeled points on the boundary of Ω: a, b, c and d in counterclockwise direction. Let us denote the boundary of Ω by $\partial\Omega$.

$\overline{\Omega} = \Omega \cup \partial\Omega$ is the topological closure of Ω, a compact subset of $\mathbb{C}$.

We will denote triangulated finite planar graphs by $\mathcal{G}$ and their Whitney-dual graph with $\mathcal{G}^*$, this means that $\mathcal{G}^*$ is a 3-regular planar graph. $\mathcal{V}$, $\mathcal{V}^*$ and $\mathcal{E}$, $\mathcal{E}^*$ denote the vertex and edge sets of $\mathcal{G}$ and $\mathcal{G}^*$, respectively.

The probability space remains as simple as possible throughout this diploma thesis (except Section 7): unifom distribution over all possible red-green colourings of the finite vertex set $\mathcal{V}$, similarly to Lemma 2.1.

Definition 2.1. ω *denotes an atomic event of this probability space, which is endowed with uniform probability measure:* $\mathbf{P}(\omega) = 2^{-|\mathcal{V}|}$. $G(\omega)$ *and* $R(\omega)$ *are the set of green, respectively, red vertices of the colouring* ω.

Definition 2.2. *Let z denote the embedding of $\mathcal{G}$ in the plane of complex numbers: for a vertex $v \in \mathcal{V}$, $z(v)$ is the complex number corresponding to that vertex.*

It is important to realize that planar graphs have several embeddings in the plane and the validity of our theorems may depend on the embedding. Thus we will try to separate geometry from graph theory. Nevertheless it is convenient to define our most important planar graph with its natural embedding.

Definition 2.3. *The third roots of unity are the complex numbers*

$$\tau = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \tau^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad \tau^3 = 1$$

We will denote the cyclic group of third roots of unity by C_3. α, β and γ will always denote generic elemens of C_3.

Definition 2.4. *The regular triangular lattice of mesh* 1 *is an infinite planar graph: the embedding of the vertex set consists of the (algebraic) lattice spanned by the third roots of unity, and two vertices are connected by an edge if their distance equals to* 1.

The regular triangular lattice of mesh $\delta > 0$ *(or briefly* $\mathcal{L}_\Delta^\delta$*) is just another embedding of the same graph: the new complex number corresponding to* $v \in \mathcal{V}$ *is* $\delta z(v)$.

This lattice is a periodic tiling of the plane with equilateral triangles. As the generalization of the triangulated rhombus of Lemma 2.1, we will consider the intersection of Ω and $\mathcal{L}_\Delta^\delta$ (see Figure 2.2). The resulting graph may not be connected, but it has a "big" connected component if δ is small. This graph $\mathcal{G}$ is the triangulation of the polygon $\partial\mathcal{G}$ which is a good approximation of the Jordan-curve $\partial\Omega$ as $\delta \to 0$. We will approximate the points a, b, c and d with vertices on $\partial\mathcal{G}$.

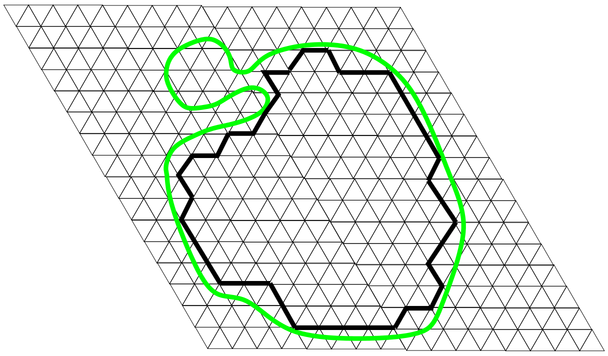

Figure 2.2: Approximating Ω with a lattice graph

Definition 2.5. *If a and b are vertices on $\partial\mathcal{G}$ then the arc connecting a to b (or briefly* $\mathsf{arc}(a,b)$*) consists of the vertices of the simple path connecting a to b on* $\partial\mathcal{G}$.

$\partial\mathcal{G}$ is a circuit of $\mathcal{G}$, so there are two different simple paths connecting a to b, but it will always be clear from the context which one we are thinking about: the path that does not contain c.

If A and B are subsets of $\mathcal{V}$ then $\{A \overset{\mathrm{G}}{\leftrightsquigarrow} B\}$ denotes the event that A is connected to B by a green simple path. Similarly, $\{A \overset{\mathrm{R}}{\leftrightsquigarrow} B\}$ is the event that A is connected to B by a red simple path.

The green cluster of A consists of those vertices of $\mathcal{V}$ that are connected to A on a green simple path. Let us remark that such a cluster is not necessarily connected, but it is a disjoint union of green connected components of $\mathcal{G}$.

Definition 2.6. $P(\delta,\Omega,a,b,c,d)$ *is the probability of the event*

$$\{\mathsf{arc}(a,b) \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(c,d)\}$$

on the triangulation of (Ω, a, b, c, d) *with* $\mathcal{L}_\Delta^\delta$.

Now we are able to formulate the conjecture which is a straightforward generalization of Lemma 2.1:

The sequence of the probabilities $P(\delta, \Omega, a, b, c, d)$ is convergent as the mesh of the lattice goes to zero, so we can define

$$P(\Omega, a, b, c, d) := \lim_{\delta \to 0} P(\delta, \Omega, a, b, c, d)$$

This conjecture implies that the limit of crossing probability is invariant under translations and rescaling: if we transform Ω with $\Phi(z) = z + c$ where $c \in \mathbb{C}$ or $\Phi(z) = r \cdot z$, where $r \in \mathbb{R}_+$ then

$$P(\Omega, a, b, c, d) = P(\Phi(\Omega), \Phi(a), \Phi(b), \Phi(c), \Phi(d)).$$

Furthermore $\Phi(z) = \beta \cdot z$ is an invariance as well for $\beta \in C_3$, because of the symmetric embedding of the regular triangular lattice. These examples suggest that rotation and dilation of the lattice of "infinitesimal" mesh will not change its percolation properties and it sounds plausible that a mapping of Ω which preserves the angles of infintesimal equilateral triangles will preserve crossing probabilities as well.

Definition 2.7. $\Phi : \Omega \to \Omega'$ *is conformal mapping of* Ω *to* Ω' *if* Φ *is analytic, injective and onto.*

Conformal mappings preserve angles locally and the derivative of a conformal function must is nonzero at any point of Ω. The inverse of Φ is a conformal mapping from Ω' to Ω, and the composition of conformal mappings is conformal. It is the theorem of Carathéodory (consult [2] or any decent book on complex functions for fundamental results about conformal mappings) that Φ can be extended to a homeomorphism between $\overline{\Omega}$ and $\overline{\Omega'}$, so $\Phi|_{\partial\Omega}$ is a homeomorphism between the Jordan-curves $\partial\Omega$ and $\partial\Omega'$. This means that the conformal equivalence of generalized rectangles is well-defined and is indeed an equivalence-relation. An equivalence class is called a conformal rectangle.

Cardy not only conjectured that

$$P(\Omega, a, b, c, d) = P(\Omega', a', b', c', d')$$

if (Ω, a, b, c, d) and $(\Omega', a', b', c', d')$ are conformally equivalent, but he gave a formula to determine the value of this limit for every conformal rectangle. He used the fundamental theorem of conformal mappings:

Theorem 2.1 (Riemann mapping theorem, [2]). *Every domain*[1] Ω *can be mapped onto the unit disc with a conformal mapping.*

[1] In this diploma thesis, Ω always denotes a domain bounded by a Jordan curve, but the Riemann mapping theorem remains valid for open, singly connected $\Omega \subsetneq \mathbb{C}$ domains.

This mapping can be extended continuously to the boundary, so the images of a, b, c and d are on the unit circle. There are many ways to map Ω onto the unit disc, but a conformal mapping of the unit disc onto itself is a linear-fractional function and preserves the cross-ratio

$$u = \frac{(d-c)(b-a)}{(c-a)(d-b)} \in [0,1]$$

of the four points. Conversely, if (a,b,c,d) and (a',b',c',d') are two four-tuples of points (in counterclockwise order) on the unit circle with the same cross-ratio, then there is a conformal transformation Φ of the unit disc such that $\Phi(a) = a'$, $\Phi(b) = b'$, $\Phi(c) = c'$, $\Phi(d) = d'$. This means that a conformal equivalence class of rectangles is characterised by u, and Cardy's formula gives the values of crossing probabilities as the function $f(u)$ of u.

$0 < u < 1$ if (a,b,c,d) are on the unit circle in counterclockwise order, and $f(u)$ satisfies the differential equation

$$u(1-u)f''(u) + \frac{2}{3}(1-2u)f'(u) = 0$$

subject to the boundary conditions $f(0) = 0$ and $f(1) = 1$. The solution is the hypergeometric function (see [1]):

$$f(u) = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2} u^{\frac{1}{3}} {}_2F_1(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, u).$$

The validity of this formula was supported by computer-based experiments (see [6]), but the conjecture defied mathematical rigor until Lennard Carleson found a strikingly simple reformulation.

The idea remains the same: we are trying to find a representative of each conformal equivalence class of rectangles for which the formula of crossing probabilities is simple. We will use an equilateral triangular domain instead of the unit disc.

Definition 2.8. *Let Δ denote the equilateral triangular domain[2] of $\mathbb{C}$ with vertices*

$$a' := 1 = \frac{1}{3} + \frac{2}{3}1, \; b' := \frac{i}{\sqrt{3}} = \frac{1}{3} + \frac{2}{3}\tau, \; c' := \frac{-i}{\sqrt{3}} = \frac{1}{3} + \frac{2}{3}\tau^2$$

Proposition 2.1. *For every domain Ω with a, b and c on $\partial\Omega$ in counterclocwise order, there exists a unique conformal mapping $\Phi : \Omega \to \Delta$ such that $\Phi(a) = a'$, $\Phi(b) = b'$ and $\Phi(c) = c'$*

Uniqueness follows from the Riemann mapping theorem and the fact that the only linear-fractional function fixing three different points is the identity function, see [2].

If we apply this mapping to a conformal rectangle (Ω, a, b, c, d) then the image of d is uniquely determined and it must lie on $\mathsf{arc}(c', a')$. Thus we conclude:

[2] We have to rotate the greek letter Δ by $-90°$ to make it look like the domain Δ!

Lemma 2.2. *$(\Omega_1, a_1, b_1, c_1, d_1)$ and $(\Omega_2, a_2, b_2, c_2, d_2)$ are conformally equivalent if and only if $d_1' = d_2'$, where $\Phi_j : (\Omega_j, a_j, b_j, c_j, d_j) \to (\Delta, a', b', c', d_j')$, $j = 1, 2$*

Proof.
$\Leftarrow$: $\Phi_2^{-1} \circ \Phi_1$ is conformal and it maps $(\Omega_1, a_1, b_1, c_1, d_1)$ to $(\Omega_2, a_2, b_2, c_2, d_2)$, so the two conformal rectangles are equivalent.
$\Rightarrow$: If $\Psi : (\Omega_1, a_1, b_1, c_1, d_1) \to (\Omega_2, a_2, b_2, c_2, d_2)$ is a conformal mapping, then

$$\Phi_2 \circ \Psi \circ \Phi_1^{-1} : (\Delta, a', b', c', d_1') \to (\Delta, a', b', c', d_2').$$

Proposition 2.1 implies that $\Phi_2 \circ \Psi \circ \Phi_1^{-1}$ must be the identity mapping, so $d_1' = \Phi_2 \circ \Psi \circ \Phi_1^{-1}(d_1') = d_2'$ $\square$

Now we are able to formulate Cardy's conjecture in Carleson's form for the critical Bernoulli site percolation on $\mathcal{L}_\Delta$. This is in fact a corollary of Theorem 4.1 and we are going to complete its proof in subsection 6.2.

Theorem 2.2 (Cardy's Conjecture in Carleson's form). *For every conformal rectangle (Ω, a, b, c, d), the limit*

$$P(\Omega, a, b, c, d) := \lim_{\delta \to 0} P(\delta, \Omega, a, b, c, d)$$

exists and it is conformally invariant:

If there exists a conformal mapping $\Phi : (\Omega, a, b, c, d) \to (\Omega', a', b', c', d')$, then

$$P(\Omega, a, b, c, d) = P(\Omega', a', b', c', d')$$

Moreover, if the domain is the equilateral triangle (Δ, a', b', c') with $d' \in \mathsf{arc}(c', a')$, then

$$P(\Delta, a', b', c', d') = \mathrm{Re}(d')$$

This means that the crossing probability is directly proportional to the length of the line segment connecting c' to d'. The theorem (together with Lemma 2.2) gives a complete characterization of the invariant mappings: Two rectangles are conformally equivalent if and only if their crossing probabilities coincide.

The very simple formulation of the conjecture will help us to reveal even deeper relations between conformally invariant probabilities and the unique conformal mapping of (Ω, a, b, c) to (Δ, a', b', c').

3 Topological properties of triangulated planar graphs

Similarly to theorems about Jordan curves, the topological properties of triangulated planar graphs are easy to state and are highly plausibe, but the rigorous proofs are rather cumbersome. That is the reason why we will be content with only showing some simple ideas in work. We will investigate triangulations only, although the same statements are valid for matching pairs of mosaic graphs, see [5].

We have already mentioned the self-duality property of triangulations in Lemma 2.1. Here is the topological generalization of the statement:

Proposition 3.1. *Let $\mathcal{G}$ denote a triangulated polygon with $a, b, c, d \in \partial\mathcal{G}$ in counterclockwise order. If we colour all vertices green or red, then exactly one of the following events occur:*

$$\{\mathsf{arc}(a,b) \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(c,d)\} \textit{ or } \{\mathsf{arc}(b,c) \overset{\mathrm{R}}{\leftrightsquigarrow} \mathsf{arc}(d,a)\}$$

It is clear that a path connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(c,d)$ separates $\mathsf{arc}(b,c)$ from $\mathsf{arc}(d,a)$ in the following sense:

Definition 3.1. *If A, B and C are subsets of $\mathcal{V}$, then C separates A from B if every path of $\mathcal{G}$ that connects A to B intersects C.*

The less trivial implication of Proposition 3.1 is as follows: if there is no green path connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(c,d)$, then the green cluster of $\mathsf{arc}(a,b)$ is disjoint from $\mathsf{arc}(c,d)$, and the red vertices on the outer boundary of this green cluster (together with the red vertices of $\mathsf{arc}(a,b)$) form a red connected component that connects $\mathsf{arc}(b,c)$ to $\mathsf{arc}(d,a)$, see Figure 3.1. This property that the outer red

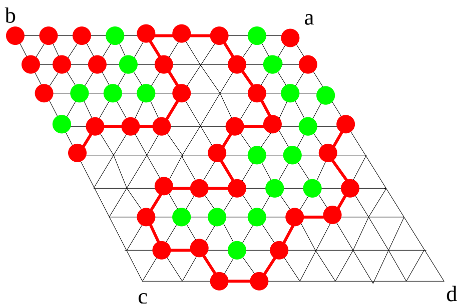

Figure 3.1: The red boundary of the green cluster of $\mathsf{arc}(a,b)$

neighbours of a green component form a red connected component that surrounds the green island will be referred to as *self-duality*, and it allows us to describe the complement of “green connected” events in terms of “red connected” events.

Self-duality is only valid for planar graphs with triangular faces: if we pick a vertex $v \in \mathcal{V}$, colour v green, colour the neighbours of v red, and all of the other vertices green, then the green connected component of v consists of v itself only, and there is a red circuit surrounding v iff the faces surrounding v are triangles.

It follows from the previous proposition that

$$\mathbf{P}(\mathsf{arc}(a,b) \overset{G}{\leftrightsquigarrow} \mathsf{arc}(c,d)) + \mathbf{P}(\mathsf{arc}(b,c) \overset{R}{\leftrightsquigarrow} \mathsf{arc}(d,a)) = 1$$

Taking into consideration that flipping colours is a measure-preserving automorphism of our probability space and assuming that the limit of crossing probabilities[3] exists as δ goes to 0, we get

$$P(\Omega,a,b,c,d) = 1 - P(\Omega,b,c,d,a). \tag{3.1}$$

It is reassuring to see that this identity does not contradict the Cardy-Carleson formula: if we consider an equilateral trianglular domain whose sides are of unit length and d is on $\mathsf{arc}(c,a)$, then the limit of crossing probabilities between $\mathsf{arc}(a,b)$ and $\mathsf{arc}(c,d)$ is the length of the interval $\mathsf{arc}(c,d)$, the limit of crossing probabilities between $\mathsf{arc}(b,c)$ and $\mathsf{arc}(d,a)$ is the length of the $\mathsf{arc}(d,a)$ interval, and the sum of these two values is indeed 1.

Another important topological property of triangulations might be called *existence and uniqueness of an extremal path.* For example, if the set of green simple paths connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(c,d)$ is non-empty then there is a unique element of this set which is the closest to $\mathsf{arc}(b,c)$ in the following sense:

Definition 3.2. *If Γ_1 and Γ_2 are paths connecting $\mathsf{arc}(a,b)$ to $\mathsf{arc}(c,d)$, then Γ_1 is closer to $\mathsf{arc}(b,c)$ than Γ_2 if Γ_1 separates the vertices of Γ_2 from $\mathsf{arc}(b,c)$.*

Instead of "closer", we should have said "not farther" because by Definition 3.1, Γ_1 is closer to $\mathsf{arc}(b,c)$ than itself.

The path that is closer to $\mathsf{arc}(c,d)$ than the others is just the green boundary of the green cluster connecting $\mathsf{arc}(a,b)$ to $\mathsf{arc}(c,d)$ (we must cut off the "dangling ends" in order to get a simple path, see Figure 3.2). The existence of an extremal path sheds new light on the Cardy-Carleson formula: if $\mathcal{G}$ is the triangulation of the domain (Δ,a',b',c') of Definition 2.8 with $\mathcal{L}_\Delta^\delta$, then there is a unique Γ_g green path connecting $\mathsf{arc}(a',b')$ to $\mathsf{arc}(a',c')$ that is the closest to $\mathsf{arc}(b',c')$[4]. Let x denote the endpoint of the random path Γ_g on $\mathsf{arc}(a',c')$.

Claim. *It follows from Theorem 2.2 that the distribution of x converges weakly to the uniform distribution on the line segment connecting a' to c' as $\delta \to 0$.*

Proof. If Γ_g' is another green path connecting $\mathsf{arc}(a',b')$ to $\mathsf{arc}(a',c')$, and x' is the endpoint of Γ_g' on $\mathsf{arc}(a',c')$, then the endpoint x of the extremal path Γ_g is closer to c' than the edpoint x' of the ordinary path Γ_g' (in the sense that $x \in \mathsf{arc}(x',c')$), so

[3] See Definition 2.6 for $P(\Omega,a,b,c,d)$.

[4] We will see later that the probability of the existence of such a green path tends to 1 as $\delta \to 0$.

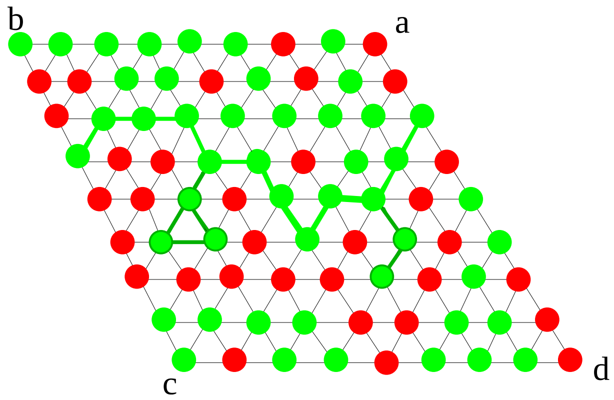

Figure 3.2: The extremal path with "dangling ends"

for an arbitrary $d' \in \mathsf{arc}(c',a')$, $\mathsf{arc}(a',b')$ is connected to $\mathsf{arc}(c',d')$ if and only if x is closer to c' than d', thus

$$\mathrm{Re}(d') = \lim_{\delta\to 0} \mathbf{P}(\mathsf{arc}(a',b') \overset{G}{\leftrightsquigarrow} \mathsf{arc}(c',d')) = \lim_{\delta\to 0} \mathbf{P}(x \in \mathsf{arc}(c',d'))$$

$\square$

Moreover, if y is the neighbour of x on $\mathsf{arc}(c',a')$ one step closer to c' than x, then y is the endpoint of Γ_r, the unique red path connecting $\mathsf{arc}(b',c')$ to $\mathsf{arc}(c',a')$ that is the closest to $\mathsf{arc}(a',b')$, because

$$\{x \in \mathsf{arc}(c',d')\} \iff \{\mathsf{arc}(a',b') \overset{G}{\leftrightsquigarrow} \mathsf{arc}(c',d')\}$$
$$\iff \{\mathsf{arc}(b',c') \overset{R}{\leftrightsquigarrow} \mathsf{arc}(d',a')\}^c \iff \{y \notin \mathsf{arc}(d',a')\}.$$

There is no general definition or theorem encapsulating the geometrical ideas behind duality or the extremal path property, but methods introduced in this section will appear in Theorem 5.1 and Lemma 6.1.

These simple topological tricks are sufficient to prove a trivial manifestation of conformal invariance.

Claim. *If (Ω,a,b,c) is a conformal triangle filled with $\mathcal{L}_\Delta^\delta$, then the probability of the following event will converge to a conformally invariant limit as $\delta \to 0$: there exists a vertex of $\mathcal{G}$ that is connected to all the three arcs on the boundary of Ω by green simple paths.*

Proof. The triviality of this claim lies in the fact that conformal invariance of a probability depending on (Ω,a,b,c) only means that the the value cannot depend on the shape of the domain, because of Proposition 2.1. We will prove that it does not depend on δ or the specific structure of $\mathcal{L}_\Delta$ either: it is $\frac{1}{2}$ for every triangulated planar graph $\mathcal{G}$.

There is a natural ordering of the green components connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(a,c)$ depending on how close they are to $\mathsf{arc}(b,c)$. Moreover, it follows from self-duality that there is a red component connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(a,c)$ in between each pair of such green components, so there is an alternating sequence of red and green components connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(a,c)$. The one that is the closest to $\mathsf{arc}(b,c)$ must be connected to it, otherwise it would have a boundary of the opposite colour: a path connecting $\mathsf{arc}(a,b)$ and $\mathsf{arc}(a,c)$, closer to $\mathsf{arc}(b,c)$ than the previous component (see Figure 3.3). This means that every percolation

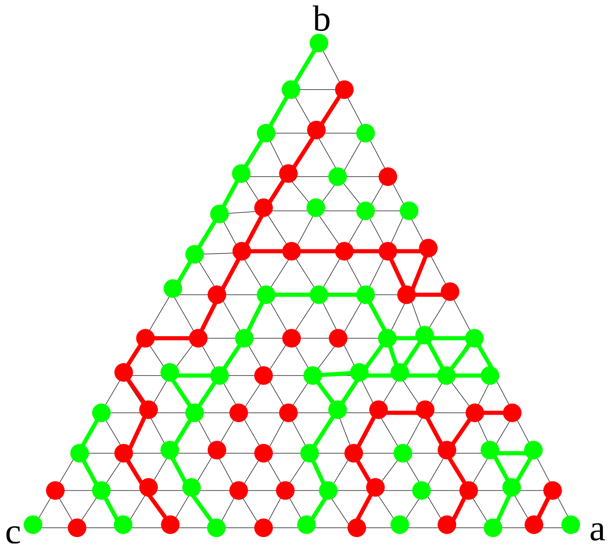

Figure 3.3: There is a red component that connects to the three boundary arcs

figure has a unique monochromatic component that is connected to all of the three boundary arcs. Flipping colours shows that two events of the same probability form a complete system of disjoint events, similarly to Lemma 2.1. □

4 Conformally invariant harmonic functions

In this section we provide the definitions necessary to formulate Smirnov's theorem and give a reformulation that links the limiting functions h_β to the conformal mapping $\Phi : \Omega \to \Delta$. In Subsection 4.2 we dicuss the properies of "harmonic conjugate triplets": although the terminology was introduced by Smirnov, he didn't clarify the definition and omitted the proof of the fact that the $\frac{2\pi}{3}$-Cauchy-Riemann equations can serve as an equivalent definition of harmonic conjugate triplets. We fill in these gaps in Lemma 4.2.

4.1 Formulation of Smirnov's theorem

Carleson's reformulation of Cardy's formula[5] already suggests Smirnov's idea of the generalization of crossing probabilities. We know from Proposition 2.1 that all "topological triangles" are conformally equivalent and we can get a system of representative elements of each equivalence class of conformal rectangles by fixing a topological triangle (Ω, a, b, c) and varying d along the arc connecting c and a (This is just a trivial generalization of Lemma 2.2). Treating d as a variable is only half of the idea: the other half is to treat it as a complex variable inside the domain Ω rather than on the boundary.

Before the definition of the generalization of crossing probabilities, let us alter our notations: from now on, $(\Omega, a(1), a(\tau), a(\tau^2))$ denotes a conformal triangle: a domain with three labeled points on the boundary in counterclockwise order. The advantage of this notation will become clear when the connection between the symmetries of the group C_3 and conformal invariance will be revealed.

Let us approximate the domain $(\Omega, a(1), a(\tau), a(\tau^2))$ with the planar graph $\mathcal{G}$ by filling it with the regular triangular lattice $\mathcal{L}_\Delta^\delta$. We approximate the points $a(\beta)$ (where $\beta \in \{1, \tau, \tau^2\}$) with vertices on $\partial\mathcal{G}$, and it will not cause any confusion if we call these vertices $a(\beta)$.

Definition 4.1. *For $z \in \overline{\Omega}$, let $Q_\beta^\delta(z)$ denote the event that there is a green simple path that connects $\mathsf{arc}(a(\beta), a(\tau\beta))$ to $\mathsf{arc}(a(\beta), a(\tau^2\beta))$ and separates z from $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$, see Figure 4.1. The probability of this event is $H_\beta^\delta(z)$.*

This definition was again a mixture of geometry and graph theory, but the two viewpoints coincide if we define the "topological path" as the curve that we get by connecting the planar embedding of the vertices of the path with line segments. If z is on the path, then $Q_\beta^\delta(z)$ occurs. Although we should keep in mind that $\mathcal{G}$ and $Q_\beta^\delta(z)$ depend on δ and the shape of $(\Omega, a(1), a(\tau), a(\tau^2))$ as well, we will omit explicit notation of this dependence.

The events $Q_\beta(z)$ are indeed a generalization of the crossing probabilities: if z is on the boundary of Ω between $a(\tau^2\beta)$ and $a(\beta)$ then a green path must connect to $\mathsf{arc}(a(\tau^2\beta), z)$ in order to separate z from $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$.

[5] See Theorem 2.2 for the Cardy-Carleson formula.

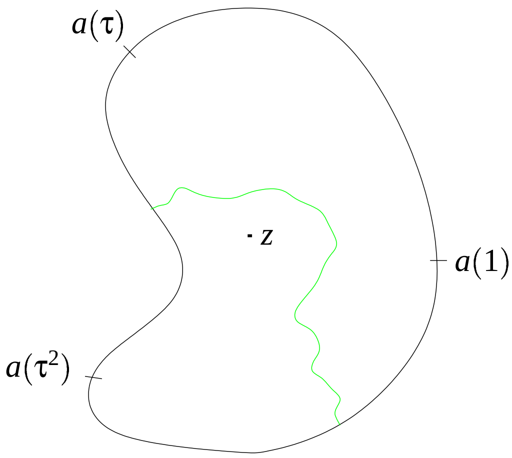

Figure 4.1: The event $Q^{\delta}_{\tau^2}(z)$ occurs

Remark.

If $z \in \mathsf{arc}(a(\beta), a(\tau^2\beta))$, *then*

$$Q_\beta(z) = \{\mathsf{arc}(a(\beta), a(\tau\beta)) \overset{G}{\leftrightsquigarrow} \mathsf{arc}(a(\tau^2\beta), z)\}.$$

If $z \in \mathsf{arc}(a(\beta), a(\tau\beta))$, *then*

$$Q_\beta(z) = \{\mathsf{arc}(z, a(\tau\beta)) \overset{G}{\leftrightsquigarrow} \mathsf{arc}(a(\tau^2\beta), a(\beta))\}.$$

If $z \in \mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$, then $\lim_{\delta\to 0} H^{\delta}_{\beta}(z) = 0$, see Lemma 5.2.

The heuristic argument behind conformal invariance was that the "distribution" of the "topological structure" of the random green subgraph of the "infinitesimal lattice" does not change as long as the mapping is angle-preserving, thus the value of the limit of $H^{\delta}_{\beta}(z)$ must be conformally invariant.

Theorem 4.1 (Smirnov's theorem, [8]). *For every domain* $(\Omega, a(1), a(\tau), a(\tau^2))$ *and every* $z \in \overline{\Omega}$, *the limit*

$$h_\beta(z) := \lim_{\delta\to 0} H^{\delta}_{\beta}(z)$$

exists, and if

$$\Phi : (\Omega, a(1), a(\tau), a(\tau^2)) \to (\Omega', a(1)', a(\tau)', a(\tau^2)')$$

is a conformal mapping with $z' = \Phi(z)$, *then*

$$h_\beta(z, \Omega, a(1), a(\tau), a(\tau^2)) = h_\beta(z', \Omega', a(1)', a(\tau)', a(\tau^2)')$$

Moreover, for the domain Δ *of Definiton* 2.8 *with vertices* $a(\beta)' = \frac{1}{3} + \frac{2}{3}\beta$, *the following formula holds:*

$$h_1(z) = \mathrm{Re}(z)$$

Corollary 4.1. *Theorem 2.2 is a special case of this theorem, because the previous remark implies that for* $z \in \mathsf{arc}(a(\tau^2)', a(1)')$,

$$h_1(z, \Delta, a(1)', a(\tau)', a(\tau^2)') = P(\Delta, a(1)', a(\tau)', a(\tau^2)', z)$$

Corollary 4.2. *If* $(\Omega, a(1), a(\tau), a(\tau^2))$ *is an equilateral triangular domain, then the function* $h_\beta(z)$ *is directly proportional to the distance of* z *and* $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$.

The second corollary formally follows from the relabeling $a(\alpha) \to a(\beta\alpha)$ of the boundary points and conformal invariance.

Thus $h_1(z) + h_\tau(z) + h_{\tau^2}(z) = 1$ holds for all $z \in \Delta$, and by the conformal invariance of the functions h_β, the identity must hold for every other domain Ω as well. We will later give a separate proof of this nontrivial fact.

Let us define another important linear combination of the functions h_β:

Definition 4.2.

$$g(z, \Omega, a(1), a(\tau), a(\tau^2)) := \sum_{\beta \in C_3} (\frac{1}{3} + \frac{2}{3}\beta) h_\beta(z)$$

We can give a useful reformulation of Smirnov's theorem using g:

Lemma 4.1. *The following two statements are equivalent:*

1. *The functions* h_β *satisfy the conditions of Theorem 4.1.*
2. $g(z, \Omega, a(1), a(\tau), a(\tau^2))$ *is the unique conformal mapping of* $(\Omega, a(1), a(\tau), a(\tau^2))$ *to* $(\Delta, a(1)', a(\tau)', a(\tau^2)')$

Proof.
1. $\Rightarrow$ 2.: $g(z, \Delta, a(1)', a(\tau)', a(\tau^2)') = z$ easily follows from $a(\beta)' = \frac{1}{3} + \frac{2}{3}\beta$ and the fact that the functions h_β are properly rescaled distances to the sides of the triangle. If Φ is the conformal mapping of Ω to Δ, then conformal invariance of the functions h_β imply

$$g(z, \Omega, a(1), a(\tau), a(\tau^2)) = g(\Phi(z), \Phi(\Omega), \Phi(a(1)), \Phi(a(\tau)), \Phi(a(\tau^2))) =$$
$$g(\Phi(z), \Delta, a(1)', a(\tau)', a(\tau^2)') = \Phi(z)$$

2. $\Rightarrow$ 1.:It is enough to check that $h_1(z, \Omega, a(1), a(\tau), a(\tau^2)) = \mathrm{Re}(\Phi(z))$, but this follows from $\Phi(z) = g(z, \Omega, a(1), a(\tau), a(\tau^2))$ and the fact that the coefficients of h_τ and h_{τ^2} in g are imaginary numbers, so

$$\mathrm{Re}(g(z, \Omega, a(1), a(\tau), a(\tau^2))) = h_1(z, \Omega, a(1), a(\tau), a(\tau^2))$$

□

It is a trivial consequence of Theorem 4.1 and Lemma 4.1 that the functions h_β are harmonic. Smirnov proves in his paper that g is an analytic function, but it is the remarkable idea of Vincent Beffara that conformal mappings can be built from the limit of the H_β^δ probabilities.

We will prove in the forthcoming sections that the limiting function $g(z,\Omega)$ exists, and it is an analytic function that satisfies boundary conditions which guarantee that g is the unique conformal mapping of Ω to Δ.

In order to verify that g is an analytic function it is enough to check that $h_1+h_\tau+h_{\tau^2}$ and $h_1+\tau h_\tau+\tau^2 h_{\tau^2}$ are analytic functions. If we consider linear subspace of $\mathbb{C}^3$ that consists of the 3-tuples of coefficients (c_1,c_τ,c_{τ^2}) for which $\sum_\beta c_\beta h_\beta$ is analytic then $(1,1,1)$ and $(1,\tau,\tau^2)$ are going to be a linear basis of this 2-dimensional subspace. The space of the analytic coefficient vectors of $(h_1(z),h_\tau(z),$ $h_{\tau^2}(z))$ cannot be 3-dimensional, because otherwise $(1,0,0)$ would be in it, and h_1 would be a real-valued analytic function, but h_1 cannot be constant because of the Cardy-Carleson formula. (c_1,c_τ,c_{τ^2}) is a (non-constant) analytic coefficient vector of $(h_1(z),h_\tau(z),h_{\tau^2}(z))$ iff c_1, c_τ and c_{τ^2} are the vertices of an equilateral triangle in counterclockwise order, and in this case $\sum_\beta c_\beta h_\beta$ is a conformal mapping of Ω to that equilateral triangle.

4.2 Harmonic conjugate triplets

We reveal further properties of the harmonic functions h_β based on the fact that $(1,1,1)$ and $(1,\tau,\tau^2)$ are analytic coefficient vectors.

Lemma 4.2. *For $f_\beta:\Omega\to\mathbb{R}$, $\beta\in C_3$, the following four statements are equivalent:*

1. *$f_1+f_\tau+f_{\tau^2}=C$ where C is a real constant and $1f_1+\tau f_\tau+\tau^2 f_{\tau^2}$ is an analytic function on Ω*

2. *f_β is continuous for all $\beta\in C_3$ and*

$$\oint_\Gamma f_\beta(z)\mathrm{d}z=\frac{1}{\tau}\oint_\Gamma f_{\tau\beta}(z)\mathrm{d}z$$

   for any smooth, simple, closed curve Γ in the domain Ω

3. *There exists an analytic function $f:\Omega\to\mathbb{C}$ and a real constant C such that*

$$\forall\beta\ \ \mathrm{Re}\Big(\frac{1}{\beta}f(z)\Big)=f_\beta(z)-\frac{C}{3}$$

4. *f_β is continuously differentiable and for all directions ν*

$$\frac{\partial}{\partial\nu}f_\beta=\frac{\partial}{\partial\tau\nu}f_{\tau\beta}$$

Proof.

1. $\Rightarrow$ 3.: We will show that $f = \frac{2}{3}(1 f_1 + \tau f_\tau + \tau^2 f_{\tau^2})$ and C satisfy the equations of 3.

$$\mathrm{Re}(\frac{1}{\beta} f(z)) = \frac{2}{3}\mathrm{Re}(\frac{1}{\beta} f_1(z) + \frac{\tau}{\beta} f_\tau(z) + \frac{\tau^2}{\beta} f_{\tau^2}(z)) =$$
$$\frac{2}{3}(f_\beta(z) - \frac{1}{2} f_{\tau\beta}(z) - \frac{1}{2} f_{\tau^2\beta}(z)) + \frac{2}{3}\frac{1}{2}(f_1(z) + f_\tau(z) + f_{\tau^2}(z)) - \frac{1}{3}C =$$
$$f_\beta(z) - \frac{1}{3}C$$

1. $\iff$ 2.: If we define $a_\beta := \oint_\Gamma f_\beta(z)\mathrm{d}z$, then 1. is equivalent to $a_1 + a_\tau + a_{\tau^2} = 0$ and $1a_1 + \tau a_\tau + \tau^2 a_{\tau^2} = 0$, beacause a real-valued analytic function is constant, and a function is analytic iff its integral vanishes on every Γ curve. a_1, a_τ and a_{τ^2} are solutions of these equations iff $a_\beta = \beta a_1$ for all β, which is equivalent to 2.

3. $\Rightarrow$ 4.: The identity $f' = \frac{1}{\nu}\frac{\partial}{\partial\nu} f = \frac{1}{\tau\nu}\frac{\partial}{\partial\tau\nu} f$ holds for the directional derivatives of the analytic function f, so

$$\frac{\partial}{\partial\nu} f_\beta = \frac{\partial}{\partial\nu}\mathrm{Re}(\frac{1}{\beta} f) = \mathrm{Re}(\frac{1}{\beta}\frac{\partial}{\partial\nu} f) = \mathrm{Re}(\frac{1}{\tau\beta}\frac{\partial}{\partial\tau\nu} f) = \frac{\partial}{\partial\tau\nu} f_{\tau\beta}.$$

4. $\Rightarrow$ 1.: $f_c := f_1 + f_\tau + f_{\tau^2}$ is a constant function, because $\frac{\partial}{\partial 1} f_c = \frac{\partial}{\partial\tau} f_c = \frac{\partial}{\partial\tau^2} f_c$ easily follows from 4. But $\frac{\partial}{\partial 1} f_c + \frac{\partial}{\partial\tau} f_c + \frac{\partial}{\partial\tau^2} f_c = 0$ is also true, because f_c is continuously differentiable and $1 + \tau + \tau^2 = 0$. To prove that $f := \sum_\beta \beta f_\beta$ is analytic, we only need to check that $\frac{1}{\nu}\frac{\partial}{\partial\nu} f$ is independent of ν for at least two different directions ν.

$$\frac{1}{1}\frac{\partial}{\partial 1} f = \sum_\beta \beta\frac{\partial}{\partial 1} f_\beta = \sum_\beta \beta\frac{\partial}{\partial\tau} f_{\tau\beta} = \frac{1}{\tau}\sum_\beta \tau\beta\frac{\partial}{\partial\tau} f_{\tau\beta} = \frac{1}{\tau}\frac{\partial}{\partial\tau} f$$

□

It is quite self-explanatory why we call f_β, $\beta \in C_3$ *harmonic conjugate triplets*: if we replace τ by i and C_3 by C_4, the cyclic group of the fourth roots of unity, then we get the well-known equivalent definitions of analytic functions: both 1. and 3. become the definition of the real and imaginary part of an analytic function, 2. becomes Morera's theorem and the equations in 4. become the Cauchy-Riemann equations. Of course $f_1 = -f_{i^2} = \mathrm{Re}(f)$ and $f_i = -f_{i^3} = \mathrm{Im}(f)$, so two members of this quadruple are redundant.

A "historical" remark: Smirnov gave a different characterization of the functions h_β and proved that $\lim_{\delta\to 0} H_\beta^\delta = h_\beta$ by showing that they are harmonic conjugate triplets. The advantage of Smirnov's characterization is that it defines h_β without explicitly mentioning the other two siblings (as opposed to Lemma 4.1). The disadvantage is that it works only if Ω has a smooth boundary.

Let $\hat{h}_\beta(z,\Omega,a(1),a(\tau),a(\tau^2))$ denote the unique harmonic function satisfying the following mixed Dirichlet-Neumann problem:

$$\begin{cases} \hat{h}_\beta = 1 \text{ at } a(\beta),\ \hat{h}_\beta = 0 \text{ on } \mathsf{arc}(a(\tau\beta),a(\tau^2\beta)) \\ \frac{\partial}{\partial \tau\nu}\hat{h}_\beta = 0 \text{ on } \mathsf{arc}(a(\beta),a(\tau\beta)) \\ \frac{\partial}{\partial(-\tau^2\nu)}\hat{h}_\beta = 0 \text{ on } \mathsf{arc}(a(\tau^2\beta),a(\beta)) \end{cases}$$

where ν is the counterclockwise-pointing unit tangent to $\partial\Omega$.

The boundary conditions defining $\hat{h}_\beta$ are conformally invariant, so uniqueness of the solution implies that

$$\hat{h}_\beta(z,\Omega,a(1),a(\tau),a(\tau^2)) = \hat{h}_\beta(\Phi(z),\Delta,a(1)',a(\tau)',a(\tau^2)')$$

where Φ is the unique conformal mapping from Ω to Δ. It is easy to check that for an equilateral triangular domain $\hat{h}_\beta$ is the properly rescaled distance from $\mathsf{arc}(a(\tau\beta),a(\tau^2\beta))$, so it follows from conformal invariance that the solution of the mixed Dirichlet-Neumann problem is indeed h_β for every Ω with a smooth boundary.

We will follow the main ideas of Smirnov's proof, although we altered some technical details: his original method consisted of verifying that the functions h_β form a harmonic conjugate triple by showing that the 2. equivalent definition of Lemma 4.2 holds. Thus the functions h_β are harmonic according to the 3. property, they satisfy the Dirichlet boundary conditions (we will prove this and more in Lemma 5.2), and the 4. property of harmonic conjugate triplets combined with the Dirichlet boundary conditions imply that the Neumann boundary conditions are satisfied as well.

5 Compactness

In this section we present Smirnov's proof of the fact that every infinite sequence of functions of form $H^{\delta}_{\beta} : \Omega \to [0,1]$ has a uniformly convergent subsequence. If $\delta \to 0$ then the limiting function of the uniformly convergent subsequence h_β satisfies boundary conditions that are sufficient to deduce that g is the unique conformal mapping of Lemma 4.1 from the fact that g is analytic. Theorem 5.3 combines the theory of compactness with that of conformal mappings to prove that the whole sequence is uniformly convergent.

5.1 RSW-theory

Even the slightest generalization of Lemma 2.1 is surprisingly hard to prove. We only state a consequence of the famous Russo-Seymour-Welsh (RSW) theorem (see [5]) in the case of $\frac{1}{2} - \frac{1}{2}$ Bernoulli site percolation on the lattice $\mathcal{L}^{\delta}_{\Delta}$:

Proposition 5.1. *If (Ω, a, b, c, d) is a rectangle with sides parallel to the real and imaginary axis then $\exists\, p > 0$ depending only on the shape of the rectangle such that*

$$\liminf_{\delta \to 0} \mathbf{P}(\mathsf{arc}(a,b) \overset{G}{\leftrightsquigarrow} \mathsf{arc}(c,d)) \geq p$$

Of course the lower bound for the probability of a horizontal crossing is becomes worse as the horizontal side of the rectangle becomes longer compared to the vertical side.

Proposition 5.1 can be very useful combined with some geometry: we can build a "tunnel" of rectangles (see Figure 5.1) and prove that there is a green path connecting the two ends of the tunnel with probability bounded away from 0 regardless of the mesh of the lattice. If the joint probability of green crossings in the rectangles of the tunnel is bounded away from 0 then these green paths must intersect to form a longer green path connecting the two ends of the tunnel. We need an important inequality to prove this:

Definition 5.1. *An event A (on our independent Bernoulli probability space of vertex colourings) is called green-increasing*[6] *if $\omega \in A$ and $G(\omega) \subseteq G(\omega')$ imply $\omega' \in A$.*

If A is the event that the two ends of the tunnel are connected by a green path inside the tunnel, then A is increasing: turning red vertices green (inside or outside the tunnel) will not ruin that green path.

Proposition 5.2 (Harris inequality, [4]). *If A and B are green-increasing events then*

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B)$$

[6] See Definition 2.1 for $G(\omega)$.

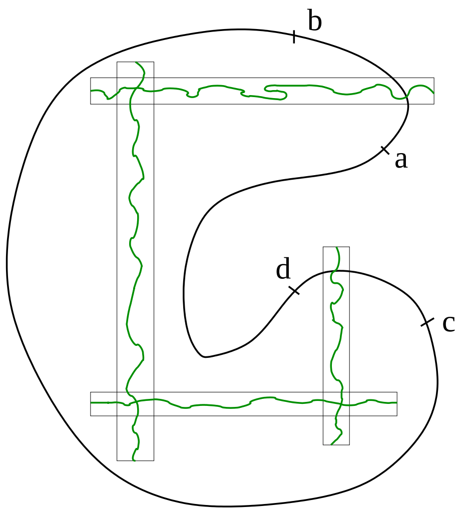

Figure 5.1: A green path connecting the two ends of the tunnel

If we iterate this inequality, we can prove that the probability of the joint occurence of these events is greater than or equal to the product of the individual probabilities.

This tunneling technique shows that the crossing probabilities are bounded away from 0 and 1 for an arbitrary domain Ω with four distinct points on the boundary: it is a geometric property of Jordan curves that there is tunnel connecting $\mathsf{arc}(a,b)$ to $\mathsf{arc}(c,d)$, and the upper bound follows from the fact that the probability of $\{\mathsf{arc}(b,c) \stackrel{\mathrm{R}}{\leftrightsquigarrow} \mathsf{arc}(d,a)\}$ is also bounded away from 0.

The next lemma gives us a bound on the tail probability of the size of the green cluster of a vertex.

Lemma 5.1. *For two concentric circles of radii* $0 < r < 1$ *and* 1*, there exists an* $\varepsilon > 0$ *and* $C > 0$ *such that for all* $\delta > 0$ *the probability of a green crossing between the circles is less than* $C \cdot r^{\varepsilon}$

Proof. To give an upper bound on the probability of a green crossing between the inner and outer boundary of this anullus, we have to show that the probability of the occurence of a red circuit separating the two circles is high.

Let us fill this anullus with several disjoint concentric anulli of fixed shape (see Figure 5.2). We can use four congruent rectangles to form an anullus whose inner and outer boundaries are both square-shaped, and Propositions 5.1 and 5.2 combined tell us that there exists a $q < 1$ such that for every δ the probability of a red circuit in the anullus is at least $1 - q$. If the bigger square is twice as big as the smaller one, than we can surround the smaller circle with $\log_2(\frac{1}{r}) - O(1)$ disjoint similar square-shaped anulli, all of which are inside the bigger circle. If there is a red circuit in any of these anulli, then there is no green crossing between

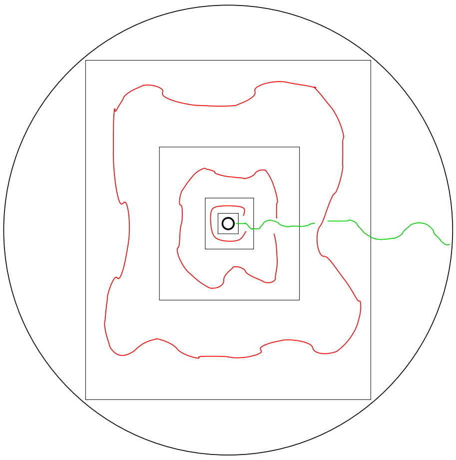

Figure 5.2: A red circuit occurs in a square-shaped anullus

the two circles. So a green crossing implies the joint occurence of $\log_2(\frac{1}{r}) - O(1)$ independent events of probability less than q.

If we define $\varepsilon := \log_2(\frac{1}{q}) > 0$, then the probability of a green crossing is less than

$$q^{\log_2(\frac{1}{r})-O(1)} = O(2^{\log_2(q)\cdot\log_2(\frac{1}{r})}) = O(r^{-\log_2(q)}) = O(r^{\varepsilon})$$

$\square$

Moreover, there is an $\hat{\varepsilon}$ and a $\hat{C} > 0$ such that the probability of a green crossing between the two circles is at least $\hat{C} \cdot r^{\hat{\varepsilon}}$ for every δ. The proof of the lower bound is similar to that of the upper bound: we have to use rectangles of fixed shape to build a “logarithmic staircase” connecting the circles.

It is believed (see [6]) that the two bounds coincide and that the order of the probability in question is $r^{\frac{5}{48}}$: in fact it is the conjecture of conformal invariance that allows statistical phisicists to determine critical exponents like this.

5.2 Uniform Equicontinuity

Let us remind the reader of Definition 4.1, where $Q_\beta^\delta(z)$ was defined for all $z \in \Omega$ in a geometrical way. $H_\beta^\delta(z)$ is a constant function as long as we vary z inside a triangular face of the lattice, so it is convenient to call this value $H_\beta^\delta(v)$, where $v \in \mathcal{V}^*$ is the vertex of $\mathcal{G}^*$ corresponding to that face of $\mathcal{G}$ in the Whitney-dual sense.

Our planar graphs are not oriented, but we can orient the edges of $\mathcal{G}$ and $\mathcal{G}^*$ in both directions:

Definition 5.2. *Let $\vec{\mathcal{E}}$ and $\vec{\mathcal{E}}^*$ denote the set of oriented edges of $\mathcal{G}$ and $\mathcal{G}^*$, respectively. For an oriented edge e, $I(e)$ denotes the intitial vertex of e and $T(e)$ denotes the terminal vertex of e. Let us denote the reverse of e by $-e$, so $I(e) = T(-e)$ and $T(e) = I(-e)$.*

If v is a vertex of the dual graph $\mathcal{G}^*$, then v^* denotes the triangular face of $\mathcal{G}$ that corresponds to v in the Whitney-dual sense. The dual correspondance between the vertices of $\mathcal{G}^*$ and the faces of $\mathcal{G}$ can be extended to the oriented edges of $\mathcal{G}^*$ and $\mathcal{G}$: if $e \in \vec{\mathcal{E}}^*$ is an oriented edge of $\mathcal{G}^*$, then $e^* \in \vec{\mathcal{E}}$ crosses e from right to left. More precisely:

Definition 5.3. *For an edge $e \in \vec{\mathcal{E}}^*$, e^* is the common edge of the triangles $I(e)^*$ and $T(e)^*$, and e^* is oriented according to the counterclockwise directon of the triangle $I(e)^*$.*

With these definitions $(-e)^* = -(e^*)$.

Definition 5.4. *If $e \in \vec{\mathcal{E}}^*$, denote by $P_\beta(e)$ the probability of the event*

$$Q_\beta(T(e)) \setminus Q_\beta(I(e)).$$

In words: $I(e)^*$ and $T(e)^*$ are adjacent triangular faces of $\mathcal{G}$, and $P_\beta(e)$ is the probability of the following event: there is a simple green path of $\mathcal{G}$ that connects $\mathsf{arc}(a(\beta), a(\tau\beta))$ to $\mathsf{arc}(a(\beta), a(\tau^2\beta))$ and separates $T(e)$ from $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$, but there is no such green path for $I(e)$.

The probabilities $P_\beta(e)$ will play an important role in the theory of conformal invariance. We will prove that $P_\beta(e) \to 0$ uniformly as $\delta \to 0$ and show in Lemma 6.1 that the values of $P_\beta(e)$ satisfy "discrete Cauchy-Riemann-equations". But first of all, let us relate the values of $P_\beta(e)$ to the discrete directional derivative of H_β^δ in the direction of the edge e:

$$\begin{aligned} H_\beta(T(e)) - H_\beta(I(e)) = \mathbf{P}(Q_\beta(T(e))) - \mathbf{P}(Q_\beta(I(e))) = \\ \big(\mathbf{P}(Q_\beta(T(e)) \setminus Q_\beta(I(e)) + \mathbf{P}(Q_\beta(T(e)) \cap Q_\beta(I(e)))\big) - \\ \big(\mathbf{P}(Q_\beta(I(e)) \setminus Q_\beta(T(e)) + \mathbf{P}(Q_\beta(I(e)) \cap Q_\beta(T(e)))\big) = \\ = P_\beta(e) - P_\beta(-e) \end{aligned} \tag{5.1}$$

Our final goal is to prove that the functions H_β^δ converge to a conformally invariant limit as the mesh of the lattice goes to 0, but there is no efficient method to compare the values of $H_\beta^\delta(z)$ for different values of δ. To get a grip on this problem, we will first prove that every infinite subset of the functions H_β^δ has a cluster point with respect to the topology defined by the supremum-norm. This is in fact equivalent to proving that the topological closure of the set of functions H_β^δ form a *compact* subset of the functions $\overline{\Omega} \to \mathbb{R}$ in the sup-norm.

Theorem (Arzela-Ascoli-theorem, [7]). *A family of continuous functions mapping $\overline{\Omega}$ to $\mathbb{R}$ form a precompact set with respect to the supremum-norm if and only if they are uniformly bounded and uniformly equicontinuous.*

To prove uniform equicontinuity, we need a technical result about Jordan-curves (see [2]), which essentially follows from the fact that a Jordan-curve itself is a uniformly continuous function mapping the unit circle into $\mathbb{R}^2$.

Proposition. *For every Ω, there is a function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that* $\lim_{x\to 0} r(x) = 0$ *and for every $z_1, z_2 \in \overline{\Omega}$, there is a curve in Ω connecting z_1 to z_2 lying inside a circle of radius $r(|z_1 - z_2|)$.*

We need the function r of this proposition to formulate the theorem which tells us that we can apply Arzela-Ascoli in the case of the functions H_β^δ.

Theorem 5.1. *For every $\Omega \subset \mathbb{C}$ there is a positive exponent ε and a finite constant C such that for all $\beta \in C_3$, for all $\delta > 0$ and for all $z_1, z_2 \in \overline{\Omega}$*

$$\left| H_\beta^\delta(z_1) - H_\beta^\delta(z_2) \right| \leq C \cdot r(|z_1 - z_2|)^\varepsilon \tag{5.2}$$

First, let us point out a slight technical problem: if the values of $H_\beta^\delta(z)$ are defined to be constant on the triangular faces of $\mathcal{G}$, then they will not even be continuous. When we prove uniform equicontinuity, we will think about the functions H_β^δ as being defined on $\mathcal{V}^*$. Using elementary interpolation, we can define a new function $H_\beta^\delta(z)$ for every $z \in \overline{\Omega}$ that is uniformly continuous with the ε, C, and r of the theorem, and the sup-difference between the old and new functions $H_\beta^\delta(z)$ is $O(\delta^\varepsilon)$ (see the Corollary of Lemma 6.1).

Proof. For the sake of definiteness we prove the theorem for $\beta = 1$. We know that H_1 is a bounded function, since its values are probabilities. So if we want to prove uniform equicontinuity, we can choose a small positive constant and prove the inequality only when $|z_1 - z_2|$ is smaller than that constant. That is the reason why we can assume that there is an $\alpha \in C_3$ such that both z_1 and z_2 are further from $\mathsf{arc}(a(\alpha), a(\tau\alpha))$ than a positive constant depending on the shape and size of Ω.

Using a similar argument as in equation (5.1), we can see that

$$H_1^\delta(z_1) - H_1^\delta(z_2) = \mathbf{P}(Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)) - \mathbf{P}(Q_1^\delta(z_2) \setminus Q_1^\delta(z_1))$$

In order to prove (5.2), it is enough to show that

$$\mathbf{P}(Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)) \leq \frac{C}{2} r(|z_1 - z_2|)^\varepsilon.$$

Let Γ_g denote the extremal green path connecting $\mathsf{arc}(a(1), a(\tau))$ to $\mathsf{arc}(a(1), a(\tau^2))$ that is the closest[7] to $\mathsf{arc}(a(\tau), a(\tau^2))$. If $Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)$ occurs then Γ_g separates z_1 from z_2. This means that the curve of the previous proposition must intersect Γ_g (see Figure 5.3).

[7] See Definition 3.2 for the extremal path.

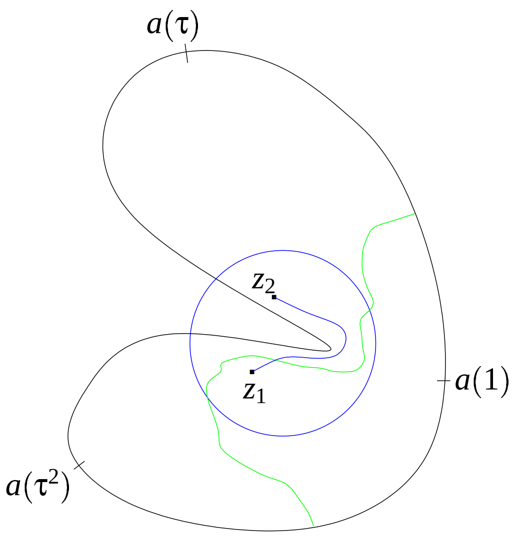

Figure 5.3: The green path Γ_g intersects the circle $\mathcal{C}$.

If $\mathcal{C}$ denotes the circle of radius $r(|z_1 - z_2|)$ containing the curve connecting z_1 and z_2, then the events $\{\mathcal{C} \overset{G}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau))\}$ and $\{\mathcal{C} \overset{G}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau^2))\}$ must occur.

Moreover $\{\mathcal{C} \overset{R}{\leftrightsquigarrow} \mathsf{arc}(a(\tau), a(\tau^2))\}$ occurs, because the extremal path Γ_g is the subset of the boundary of the red cluster of $\mathsf{arc}(a(\tau), a(\tau^2))$ (see Figure 3.2), so the red cluster of $\mathsf{arc}(a(\tau), a(\tau^2))$ must have a point inside $\mathcal{C}$ (otherwise Γ_g would be disjoint from $\mathcal{C}$ and could not separate z_1 from z_2).

Particularly, there is a monochromatic path connecting $\mathcal{C}$ to $\mathsf{arc}(a(\alpha), a(\tau\alpha))$. But the circle contains z_1, so it is further from the arc than a positive constant depending on Ω only. A circle of radius $r(|z_1 - z_2|)$ is connected to a circle of radius $O(1)$ by a monochromatic cluster, so we can apply Lemma 5.1 to prove that the event $Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)$ implies the occurence of an event of probability less than $C \cdot r(|z_1 - z_2|)^\varepsilon$. $\square$

We can get rid of the mysterious function r when the straight line segment connecting z_1 and z_2 lies entirely in the domain Ω. For example, when Ω is convex, $r(x) = \frac{x}{2}$ is enough, which implies the Hölder-continuity of the functions H_β^δ.

Compactness allows us to speak about limits: if we consider an ifinite sequence of functions of form H_β^δ, then this sequence has a convergent subsequence in the Banach space of continuous $\overline{\Omega} \to [0, 1]$ functions. Thus we can assume that there is a δ_n infinite sequence with $\lim_{n\to\infty} \delta_n = 0$ such that the uniform limit

$$h_\beta = \lim_{n\to\infty} H_\beta^{\delta_n}$$

exists for all $\beta \in C_3$. As the first step on our way to prove that the function g of Definition 4.2 is the unique conformal mapping of Ω to the equilateral triangle Δ

of Definition 2.8, we will investigate the boundary behaviour of g.

Lemma 5.2. *The function*

$$g(z) = \sum_{\beta \in C_3} (\frac{1}{3} + \frac{2}{3}\beta) h_\beta(z) = \sum_\beta a(\beta)' h_\beta(z)$$

maps $\partial\Omega$ homeomorphically to the boundary of Δ, and for all β

$$g(a(\beta)) = a(\beta)' = \frac{1}{3} + \frac{2}{3}\beta$$

Proof. It makes sense to talk about the boundary values of the functions H^δ_β, because of uniform equicontinuity, although these functions are only defined inside the polygon $\partial\mathcal{G}$, but this polygon approximates $\partial\Omega$ well as $\delta \to 0$. Moreover, let us assume that $a(\beta)$ is a vertex of $\partial\mathcal{G}$.

In order to prove $g(a(\beta)) = a(\beta)'$, we only need to check that $h_\beta(a(\beta)) = 1$ and $h_{\tau\beta}(a(\beta)) = h_{\tau^2\beta}(a(\beta)) = 0$.

$H^\delta_\beta(a(\beta)) = 1 - O(\delta^\varepsilon)$, because self-duality implies that there is a green path connecting $\mathsf{arc}(a(\beta), a(\tau\beta))$ to $\mathsf{arc}(a(\beta), a(\tau^2\beta))$ if and only if the red component of $a(\beta)$ does not reach $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$, so

$$1 - H^\delta_\beta(a(\beta)) = \mathbf{P}(a(\beta) \overset{\text{R}}{\leftrightsquigarrow} \mathsf{arc}(a(\tau\beta), a(\tau^2\beta))) = O(\delta^\varepsilon)$$

is a consequence of Lemma 5.1 and the fact that there is a circle of positive radius around $a(\beta)$ that is disjoint from $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$. So $h_\beta(a(\beta)) = 1$ follows from uniform convergence.

$H^\delta_\beta(z) = O(\delta^\varepsilon)$ if z is on the arc connecting the vertices $a(\tau\beta)$ and $a(\tau^2\beta)$, because if there is a green path that separates z from $\mathsf{arc}(a(\tau\beta), a(\tau^2\beta))$, then this path must contain z, so

$$H^\delta_\beta(z) \leq \min\left\{\mathbf{P}\left(z \overset{\text{G}}{\leftrightsquigarrow} \mathsf{arc}(a(\beta), a(\tau\beta))\right), \mathbf{P}\left(z \overset{\text{G}}{\leftrightsquigarrow} \mathsf{arc}(a(\beta), a(\tau^2\beta))\right)\right\} = O(\delta^\varepsilon)$$

Thus we established $g(a(\beta)) = a(\beta)'$, and we are going to show that points on $\mathsf{arc}(a(\beta), a(\tau\beta))$ are mapped to points on the line segment connecting $a(\beta)'$ to $a(\tau\beta)'$. For the sake of definiteness, let us assume that $\beta = 1$.

$h_{\tau^2}(z) = 0$ for a point z on $\mathsf{arc}(a(1), a(\tau))$, so it is enough to show that $h_1(z) + h_\tau(z) = 1$ holds, because then $g(z)$ is the convex combination of the values $a(1)'$ and $a(\tau)'$ with coefficients $h_1(z)$ and $h_\tau(z)$. The desired identity is a consequence of equation (3.1) and the Remark after Definition 4.1.

It follows from the continuity of g that $\mathsf{arc}(a(\beta), a(\tau\beta))$ is mapped onto the arc connecting $a(\beta)'$ to $a(\tau\beta)'$ on the boundary of Δ. Both boundaries are compact subsets of $\mathbb{C}$, so we only need to check that g maps different points of $\partial\Omega$ to different points of the boundary of Δ in order to prove that g is homeomorphism between these arcs.

It is enough to show that the value of $h_1(z)$ is strictly decreasing on $\mathsf{arc}(a(1),a(\tau))$ as z is getting closer to $a(\tau)$: if both z_1 and z_2 are on $\mathsf{arc}(a(1),a(\tau))$, and $z_1 \in \mathsf{arc}(a(1),z_2)$, then we need to check that it is easier to separate z_1 than z_2 from $\mathsf{arc}(a(\tau),a(\tau^2))$: $h_1(z_1) > h_1(z_2)$.

We can build two disjoint tunnels[8] of rectangles connecting $\mathsf{arc}(z_1,z_2)$ to $\mathsf{arc}(a(1),a(\tau^2))$ and to $\mathsf{arc}(a(\tau),a(\tau^2))$ respecitvely (see Figure 5.4).

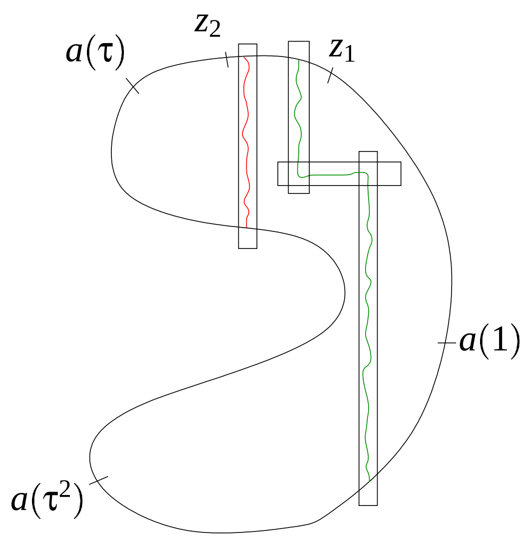

Figure 5.4: $\lim_{\delta\to 0} \mathbf{P}(Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)) > 0$

Using Propositions 5.1 and 5.2 and the independence of events happening on the two disjoint tunnels, we can see that the liminf of the probability of the simultaneous occurence of a green path connecting $\mathsf{arc}(z_1,z_2)$ to $\mathsf{arc}(a(1),a(\tau^2))$ and a red path connecting $\mathsf{arc}(z_1,z_2)$ to $\mathsf{arc}(a(\tau),a(\tau^2))$ is bounded away from 0. This means that

$$0 < \lim_{\delta\to 0} \mathbf{P}(Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)) = \lim_{\delta\to 0} \left(H_1^\delta(z_1) - H_1^\delta(z_2)\right) = h_1(z_1) - h_1(z_2)$$

$\square$

Proposition 5.3. *If $f : \overline{\Omega} \to \mathbb{C}$ is analytic on Ω, continuous on $\overline{\Omega}$ and maps the boundary of Ω bijectively to the boundary of Ω', then f is conformal.*

The proof essentially follows from the argument principle: since $f(\partial\Omega)$ is a Jordan-curve, it winds around the points of Ω' once, so the number of the solutions of $f(z) = z'$ is exactly one if z' is inside Ω'. Similarly, no point inside Ω is mapped outside $\partial\Omega'$, because the curve doesn't wind around such points. This means that f is a bijection between the points of Ω and Ω', so it is a conformal mapping.

[8]See Subsection 5.1 for the definition of a tunnel.

Let us consider a triple sequence of functions $H_\beta^\delta(z)$ defined by a sequence of regular triangular lattices with mesh tending to 0 on a domain $(\Omega, a(1), a(\tau), a(\tau^2))$. It is convenient to use the continuous, interpolated versions of these functions, described in Theorem 5.1.

Appropriate linear combinations of the functions $H_\beta^\delta(z)$ "almost satisfy" the criteria of Morera's theorem if δ is small:

Theorem 5.2. *For any simple, closed, smooth curve* Γ *lying entirely inside the domain* Ω*,*

$$\oint_\Gamma H_1(z) + H_\tau(z) + H_{\tau^2}(z)\mathrm{d}z = O(l\delta^\varepsilon)$$

and

$$\oint_\Gamma 1H_1(z) + \tau H_\tau(z) + \tau^2 H_{\tau^2}(z)\mathrm{d}z = O(l\delta^\varepsilon)$$

where ε *is defined in Lemma 5.1 and* l *is the length of* Γ*.*

We provide the proof of this theorem later, in Subsection 6.2.

Now we have the ingredients to finish the proof of Smirnov's theorem:

Theorem 5.3. *For all* $\beta \in C_3$*, the uniform limits* $h_\beta(z) = \lim_{\delta\to 0} H_\beta^\delta(z)$ *exist and the function* $g(z, \Omega, a(1), a(\tau), a(\tau^2))$ *of Definition 4.2 is the unique conformal mapping of* $(\Omega, a(1), a(\tau), a(\tau^2))$ *to the equilateral triangular domain* $(\Delta, a(1)', a(\tau)', a(\tau^2)')$ *of Definition 2.8.*

Proof. Let us define $H^\delta : \overline{\Omega} \to \mathbb{R}^3$ in such a way that the coordinates of $H^\delta(z)$ are $H_1^\delta(z)$, $H_\tau^\delta(z)$, and $H_{\tau^2}^\delta(z)$. It follows from the Arzela-Ascoli theorem and Theorem 5.1 that the functions $\{H^\delta : \delta > 0\}$ form an infinite precompact subset of the Banach space of continuous functions mapping $\overline{\Omega}$ to $\mathbb{R}^3$ with respect to sup-norm. We prove that this set has a unique cluster point: the function h, whose coordinate functions satisfy $h_1 + h_\tau + h_{\tau^2} = 1$ and $\sum_\beta a(\beta)' h_\beta$ is the unique conformal mapping of Ω to Δ. These equations determine h uniquely, because the vectors $(1,1,1)$, $(\mathrm{Re}(1), \mathrm{Re}(\tau), \mathrm{Re}(\tau^2))$ and $(\mathrm{Im}(1), \mathrm{Im}(\tau), \mathrm{Im}(\tau^2))$ are independent over $\mathbb{R}$.

We need to check that any convergent subsequence of the functions H^δ converges to h. If $\lim_{\delta\to 0} H^\delta = \hat{h}$, then the boundary behaviour of $\hat{h}$ is characterized by Lemma 5.2, moreover

$$\oint_\Gamma \sum_\beta \beta \hat{h}_\beta(z)\mathrm{d}z = \lim_{\delta\to 0} \oint_\Gamma \sum_\beta \beta H_\beta^\delta(z)\mathrm{d}z = \lim_{\delta\to 0} O(l\delta^\varepsilon) = 0$$

for any Γ. The contour integrals vanish, so $\sum_\beta \beta\hat{h}_\beta(z)$ is analytic according to Morera's theorem. Similarly, $\sum_\beta \hat{h}_\beta(z)$ is a real-valued analytic function, so it must be constant. The value of the constant is determined by the boundary behaviour of $\hat{h}$.

$\hat{g} = \sum_\beta a(\beta)' \hat{h}_\beta$ is a linear combination of analytic functions, so $\hat{g}$ is analytic as well, and putting Lemma 5.2, Proposition 5.3, and Proposition 2.1 together, we get that $\hat{g}$ is the unique conformal mapping of the domain $(\Omega, a(1), a(\tau), a(\tau^2))$ to the domain $(\Delta, a(1)', a(\tau)', a(\tau^2)')$. Thus $\hat{h} = h$, which implies that the whole sequence of the functions H^δ converges to h as δ goes to 0. $\square$

Arguments based on the compactness of the Banach space of continous functions make the theorems about the approximation of the function g simple and clear, but we cannot deduce any information about the rate of convergence from these abstract arguments. Nevertheless, the careful formulation of the order of the error term in Theorem 5.2 turns out to be useful if we want to obtain better bounds: Smirnov states in his paper that

$$h_\beta(z) = H^\delta_\beta(z) + O(\delta^\varepsilon)$$

follows directly from the analyticity conditions of Theorem 5.2, the boundary conditions of Lemma 5.2 and the theory of complex functions. He also states that the true value of ε is $\frac{2}{3}$, and it is important to emphasize the reason why this critical exponent is different from that of Lemma 5.1: $\frac{5}{48}$ is the critical exponent of the probability of a monochromatic path connecting the inner and outer boundary of an anullus, while $\frac{2}{3}$ is the exponent of three disjoint paths (not all of the same colour) connecting the inner and outer boundaries of the anullus .

6 Approximately analytic functions on the lattice

We have already seen in the previous sections that we need an approximate form of analyticity of functions related to the events Q_β^δ to prove Smirnov's theorem about the conformal invariance of the limiting probability of these events. The theorems about the contour integrals of the functions H_β^δ are in fact statements about discrete Riemann sums, and the fact that the sums are of small order follows from an identity that can be regarded as the discrete version of the 4. equivalent definition of harmonic conjugate triplets in Lemma 4.2.

6.1 Discrete Cauchy-Riemann equations

The idea that the functions h_β are harmonic conjugate triplets whose "mother function" (see the 3. equivalent definition of Lemma 4.2) is a conformal mapping to an equilateral triangle suggests that conformal invariance has a connection with the rotational symmetries of the group[9] C_3. Stanislav Smirnov managed to find the discrete counterparts of these triple symmetries, and called them the discrete Cauchy-Riemann-equations. Although we will use the Cauchy-Riemann equations to prove that certain linear combinations of the functions H_β^δ converge to an analytic limit, they are purely combinatorical identities and do not depend on the embedding of the graph in the complex plane.

Let the group C_3 act on the oriented edges of the 3-regular planar graph $\mathcal{G}^*$ in the folowing way:

Definition 6.1. *If $e \in \vec{\mathcal{E}}^*$, let $\tau(e) \in \vec{\mathcal{E}}^*$ denote the unique edge satisfying*[10] $I(\tau(e)) = I(e)$ *and* $I(\tau(e)^*) = T(e^*)$*, see Figure 6.1.*

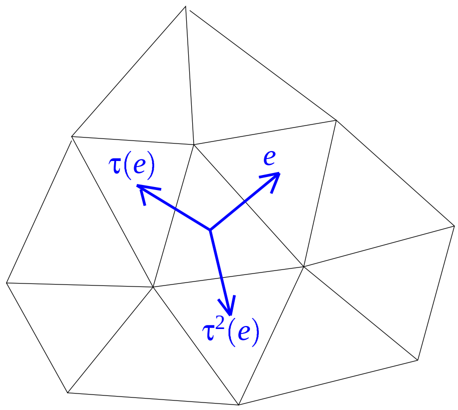

Figure 6.1: The group C_3 acts on $\vec{\mathcal{E}}^*$

[9] See Definition 2.3 for C_3.

[10] See Definition 5.2 for $I(e)$ and $T(e)$ and Definition 5.3 for e^*.

If we consider the symmetric embedding of the regular triangular lattice then $\mathcal{G}^*$ is the regular hexagonal lattice and τ rotates an edge of $\vec{\mathcal{E}}^*$ with $\frac{2\pi}{3}$ to the left, τ^2 rotates it to the right and $\tau^3 = 1$ is the identity.

If $v \in \mathcal{V}^*$ and e_v is one of the three oriented edges with initial vertex v, then the orbit of e_v under the action of C_3 is

$$\{\beta(e_v) : \beta \in C_3\} = \{e : I(e) = v\} \tag{6.1}$$

Lemma 6.1 (Discrete $\frac{2\pi}{3}$-Cauchy-Riemann equations). *For any triangulated domain $(\Omega, a(1), a(\tau), a(\tau^2))$, any oriented edge*[11] *$e \in \vec{\mathcal{E}}^*$ and any $\beta \in C_3$:*

$$P_\beta(e) = P_{\tau\beta}(\tau(e))$$

Proof. For the sake of definiteness, we will prove the case of $\beta = 1$. $P_1(e)$ is the probability of the event $Q = Q_1(T(e)) \setminus Q_1(I(e))$.

If $\omega \in Q$ is a particular colouring, then there is a green simple path Γ_g that connects $\mathsf{arc}(a(1), a(\tau))$ to $\mathsf{arc}(a(1), a(\tau^2))$ and separates $T(e)$ from $\mathsf{arc}(a(\tau), a(\tau^2))$, but there is no such green path for $I(e)$, so Γ_g must go in between $I(e)$ and $T(e)$.

The triangle $I(e)^*$ has three vertices in $\mathcal{V}$: let us denote these verices by X, Y, and Z in counterclockwise order, and $Y = I(e^*)$, $Z = T(e^*)$.

If $\omega \in Q$ then $\omega \in \{Z \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau))\}$ and $\omega \in \{Y \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau^2))\}$, and these two halves of Γ_g are vertex-disjoint. Furthermore

$$\omega \in \{X \overset{\mathrm{R}}{\leftrightsquigarrow} \mathsf{arc}(a(\tau), a(\tau^2))\},$$

because Y and Z are on the green path Γ_g and X is not, so the colour of X is red and the red cluster of X must be connected to $\mathsf{arc}(a(\tau), a(\tau^2))$, because otherwise the green boundary of this cluster (together with Γ_g) would form a green simple path that connects $\mathsf{arc}(a(1), a(\tau))$ to $\mathsf{arc}(a(1), a(\tau^2))$ and separates $I(e)$ from $\mathsf{arc}(a(\tau), a(\tau^2))$, similarly to Lemma 3.1.

This red path is of course vertex disjoint from the previous two green paths (see Figure 6.2). Thus we conclude that $Q = Q_{rgg}$ where

$$Q_{rgg} = \{X \overset{\mathrm{R}}{\leftrightsquigarrow} \mathsf{arc}(a(\tau), a(\tau^2)), Y \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau^2)), \\ Z \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau)) \text{ on disjoint paths}\}$$

Similarly, $P_\tau(\tau(e))$ is the probability of the event

$$Q_{grg} = \{X \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(a(\tau), a(\tau^2)), Y \overset{\mathrm{R}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau^2)), \\ Z \overset{\mathrm{G}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau)) \text{ on disjoint paths}\}$$

The "rotational symmetry" of the probability of these events follows from the topological properties of triangulated planar graphs and the fact that the measure

[11] See Definition 5.4 for $P_\beta(e)$.

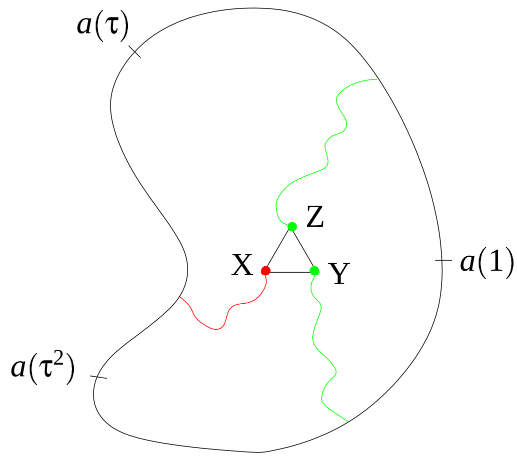

Figure 6.2: The event Q_{rgg}

on our probability space is the normed counting measure on the set of red-green vertex colourings of $\mathcal{V}$. We will show a bijection between the atomic events of Q_{rgg} and Q_{grg}. In fact, we will show a bijection Φ between the atomic events of Q_{rgg} and the event

$$Q_{rgr} = \{X \overset{\text{R}}{\leftrightsquigarrow} \mathsf{arc}(a(\tau), a(\tau^2)), Y \overset{\text{G}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau^2)),$$
$$Z \overset{\text{R}}{\leftrightsquigarrow} \mathsf{arc}(a(1), a(\tau)) \text{ on disjoint paths}\}$$

but we obtain a trivial bijection between Q_{grg} and Q_{rgr} by flipping colours.

Let us construct $\Phi : Q_{rgg} \to Q_{rgr}$. If $\omega \in Q_{rgg}$ then there is a unique counterclockwise-most extremal red path Γ'_r connecting X to $\mathsf{arc}(a(\tau), a(\tau^2))$ and a clockwise-most extremal green path Γ'_g connecting Y to $\mathsf{arc}(a(1), a(\tau^2))$. Note that it is necessary for the definition of these extremal paths that there is no monochromatic circuit surrounding $I(e)^*$.

The concatenation of Γ'_r and Γ'_g is a simple path Γ that connects $\mathsf{arc}(a(\tau), a(\tau^2))$ to $\mathsf{arc}(a(1), a(\tau^2))$. Let us define the red-green colouring of $\Phi(\omega)$ in the following way: flip the colours of those vertices of $\mathcal{V}$ that are not separated from $\mathsf{arc}(a(1), a(\tau))$ by Γ (See Figure 6.3). So we don't flip X and Y (and the other points on Γ), but flip Z, since the three points are connected to the three boundary arcs of Ω on disjoint simple paths.

The counterclockwise-most red path connecting X to $\mathsf{arc}(a(\tau), a(\tau^2))$ remains Γ'_r and the clockwise-most green path connecting Y to $\mathsf{arc}(a(1), a(\tau^2))$ remains Γ'_g on the colouring $\Phi(\omega)$, because the fact that they are extremal depends only on the colour of the set of vertices that are separated from $\mathsf{arc}(a(1), a(\tau))$ by Γ, and these are exactly the vertices that Φ didn't flip. But the green path connecting Z to $\mathsf{arc}(a(1), a(\tau))$ changes its colour to red. So $\Phi(\omega) \in Q_{rgr}$, and it is trivial to see that Φ has a well-defined inverse (flipping again) on Q_{rgr}. Thus the cardinality of the

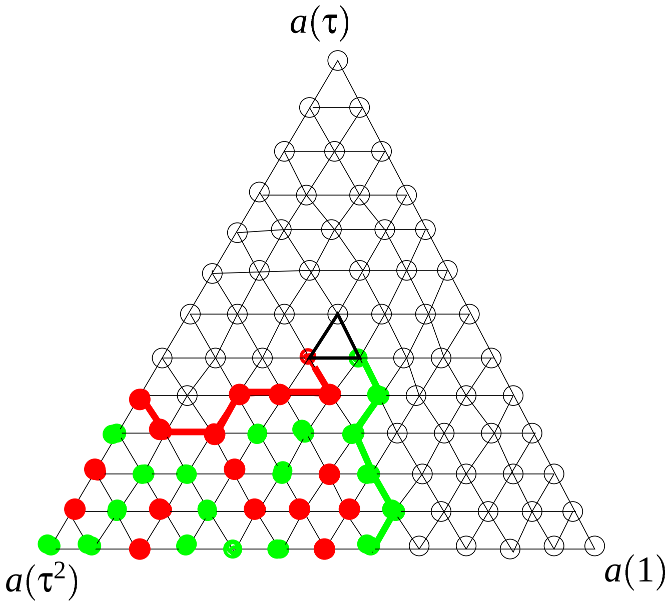

Figure 6.3: Φ flips the vertices marked by empty circles

atomic events of Q_{rgg} and Q_{rgr} are the same, which implies $P_1(e) = P_\tau(\tau(e))$. $\square$

If we combine the arguments of this lemma with those of Lemma 5.1 and Theorem 5.1, we can get an estimate of the order of $P_\beta(e)$:

Corollary 6.1. *For any Ω filled with $\mathcal{L}_\Delta^\delta$ and for any oriented edge $e \in \vec{\mathcal{E}}^*$:*

$$P_\beta(e) = O(\delta^\varepsilon)$$

This estimate follows from the fact that the triangle with vertices X, Y, and Z can be surrounded by a circle $\mathcal{C}$ of radius $O(\delta)$, and $\mathcal{C}$ is connected to all the three arcs by three disjoint paths (not all of the same colour).

It is believed that the true asymptotic behaviour of this event is $P_\beta(e) = O(\delta^{\frac{2}{3}})$ if e is well away from all boundary arcs (see the end of Section 5).

6.2 Discrete contour integrals

This subsection is devoted to the proof of Theorem 5.2. We will transform the contour integral into a sum of contour integrals around lattice triangles and let the Cauchy-Riemann equations do the local cancellation.

After embedding[12] $\mathcal{G}$ in the complex plane, we can view orinted edges as vectors. For an oriented edge $e \in \vec{\mathcal{E}}^*$, the dual edge e^* is in $\vec{\mathcal{E}}$, and we define

$$\mathrm{d}z(e^*) := z(T(e^*)) - z(I(e^*)).$$

[12] See Definition 2.2 for embeddings.

Obviously $\mathrm{d}z(-e^*) = -\mathrm{d}z(e^*)$, and using identity (6.1)

$$\sum_{e:I(e)=v} \mathrm{d}z(e^*) = \sum_{\beta\in C_3} \mathrm{d}z(\beta(e_v)^*) = 0, \tag{6.2}$$

because adding up the three vectors that surround a triangle in counterclockwise orientation, we get 0.

We want to prove that the linear combinations of the functions H_β with coefficient vectors $(1,1,1)$ and $(1,\tau,\tau^2)$ "almost" satisfy the criteria of Morera's theorem. We will define a discrete approximation of $\oint_\Gamma H^\delta_\beta(z)\mathrm{d}z$, where Γ is a simple, closed, smooth curve of length l.

First, we approximate Γ with a lattice polygon Γ^δ, see Figure 2.2. If $\mathcal{D}^* \subseteq \mathcal{V}^*$ denotes the set of vertices of $\mathcal{G}^*$ surrounded by Γ^δ, then

$$\{e^* : I(e) \in \mathcal{D}^*, T(e) \notin \mathcal{D}^*\}$$

is the edge set of Γ^δ, oriented in counterclockwise direction.

For the definition of the Riemann sum approximating the contour integral of $H^\delta_\beta(z)$ on Γ, we will use $H^\delta_\beta(v)$ defined on the vertices of $\mathcal{V}^*$.

Definition.

$$\oint^\delta_{\partial\mathcal{D}^*} H_\beta(z)\mathrm{d}z := \sum_{e:I(e)\in\mathcal{D}^*, T(e)\notin\mathcal{D}^*} \frac{H_\beta(I(e)) + H_\beta(T(e))}{2}\mathrm{d}z(e^*) \tag{6.3}$$

It is easy to see that the length of Γ^δ is approximately the same as the length of Γ:

$$\sum_{e:I(e)\in\mathcal{D}^*, T(e)\notin\mathcal{D}^*} |\mathrm{d}z(e^*)| = O(l),$$

and the discrete contour integral approximates the real contour integral well:

$$\oint^\delta_{\partial\mathcal{D}^*} H_\beta(z)\mathrm{d}z = \oint_\Gamma H^\delta_\beta(z)\mathrm{d}z + O(l\delta^\varepsilon),$$

because the distance between the smooth curve Γ and Γ^δ is $O(\delta)$ and

$$z_1 - z_2 = O(\delta) \implies H^\delta_\beta(z_1) - H^\delta_\beta(z_2) = O(\delta^\varepsilon).$$

We get this estimate by combining the Corollary of Lemma 6.1, the identity (5.1) and the fact that the continous function $H^\delta_\beta(z)$ is an interpolation between the values $H^\delta_\beta(v)$ (see Theorem 5.1).

We are going to transform this sum into a form where the discrete Cauchy-Riemann-equations can be applied.

Lemma 6.2.

$$\oint^\delta_{\partial\mathcal{D}^*} H_\beta(z)\mathrm{d}z = \sum_{e:I(e)\in\mathcal{D}^*} P_\beta(e)\mathrm{d}z(e^*) + O(l\delta^\varepsilon) \tag{6.4}$$

Proof. A term in the dicrete contour integral changes sign if we reverse the edge e:

$$\frac{H_\beta(I(e))+H_\beta(T(e))}{2}\mathrm{dz}(e^*)+\frac{H_\beta(I(-e))+H_\beta(T(-e))}{2}\mathrm{dz}(-e^*)=0.$$

If we sum over all oriented edges with both endvertices in D, then

$$\sum_{e:I(e)\in\mathcal{D}^*,T(e)\in\mathcal{D}^*}\frac{H_\beta(I(e))+H_\beta(T(e))}{2}\mathrm{dz}(e^*)=0$$

because the terms of reverse edges cancel. Adding this to equation (6.3), we get

$$\oint_{\partial\mathcal{D}^*}^{\delta}H_\beta(z)\mathrm{dz}=\sum_{e:I(e)\in\mathcal{D}^*}\frac{H_\beta(I(e))+H_\beta(T(e))}{2}\mathrm{dz}(e^*)=$$
$$\sum_{v\in\mathcal{D}^*}\sum_{e:I(e)=v}\frac{H_\beta(I(e))+H_\beta(T(e))}{2}\mathrm{dz}(e^*)\qquad(6.5)$$

Summing equation (6.2) for all $v\in\mathcal{D}^*$ and e such that $I(e)=v$ with coefficients $H_\beta(I(e))=H_\beta(v)$,

$$\sum_{v\in\mathcal{D}^*}\sum_{e:I(e)=v}H_\beta(I(e))\mathrm{dz}(e^*)=0.$$

Subtracting this from equation (6.5), we get

$$\oint_{\partial\mathcal{D}^*}^{\delta}H_\beta(z)\mathrm{dz}=\sum_{v\in\mathcal{D}^*}\sum_{e:I(e)=v}\frac{H_\beta(T(e))-H_\beta(I(e))}{2}\mathrm{dz}(e^*)$$

This is a sum of the discrete derivatives of H_β, so we can use equation (5.1) to replace $H_\beta(T(e))-H_\beta(I(e))$ with $P_\beta(e)-P_\beta(-e)$.

$$\oint_{\partial\mathcal{D}^*}^{\delta}H_\beta(z)\mathrm{dz}=\sum_{e:I(e)\in\mathcal{D}^*}\frac{P_\beta(e)-P_\beta(-e)}{2}\mathrm{dz}(e^*)=$$
$$\sum_{e:I(e)\in\mathcal{D}^*}\frac{P_\beta(e)}{2}\mathrm{dz}(e^*)+\sum_{e:I(e)\in\mathcal{D}^*}\frac{P_\beta(-e)}{2}\mathrm{dz}(-e^*)=$$
$$\sum_{e:I(e)\in\mathcal{D}^*}\frac{P_\beta(e)}{2}\mathrm{dz}(e^*)+\sum_{e:T(e)\in\mathcal{D}^*}\frac{P_\beta(e)}{2}\mathrm{dz}(e^*)$$

In order to finish the proof of (6.4), we only have to show that

$$\sum_{e:I(e)\in\mathcal{D}^*}\frac{P_\beta(e)}{2}\mathrm{dz}(e^*)=\sum_{e:T(e)\in\mathcal{D}^*}\frac{P_\beta(e)}{2}\mathrm{dz}(e^*)+O(l\delta^\varepsilon)$$

The difference between the two sums is of small order because they only differ in the boundary terms, and the order of $P_\beta(e)$ is $O(\delta^\varepsilon)$:

$$\left| \sum_{e:I(e)\in\mathcal{D}^*} P_\beta(e)\mathrm{d}z(e^*) - \sum_{e:T(e)\in\mathcal{D}^*} P_\beta(e)\mathrm{d}z(e^*) \right| =$$

$$\left| \sum_{e:I(e)\in\mathcal{D}^*,T(e)\notin\mathcal{D}^*} P_\beta(e)\mathrm{d}z(e^*) - \sum_{e:T(e)\in\mathcal{D}^*,I(e)\notin\mathcal{D}^*} P_\beta(e)\mathrm{d}z(e^*) \right| =$$

$$O(\delta^\varepsilon)\left(\sum_{e:I(e)\in\mathcal{D}^*,T(e)\notin\mathcal{D}^*} |\mathrm{d}z(e^*)| \right) = O(\delta^\varepsilon)O(l)$$

□

We will first use equation (6.4) combined with Lemma 6.1 to prove that $h_1 + h_\tau + h_{\tau^2} \equiv 1$. We do not use the properties of the particular embedding of the planar graph $\mathcal{G}$, so this equation is more than conformally invariant, because it remains valid even if we “stretch” the graph in a non-conformal way. Nevertheless, the method of the proof is to show that $h_1 + h_\tau + h_{\tau^2}$ is an analytic function.

Lemma 6.3.

$$\oint_{\partial\mathcal{D}^*}^{\delta} \sum_{\beta\in C_3} H_\beta(z)\mathrm{d}z = O(l\delta^\varepsilon)$$

Proof. We transform the integral using the previous lemma:

$$\oint_{\partial\mathcal{D}^*}^{\delta} \sum_{\beta\in C_3} H_\beta(z)\mathrm{d}z = \sum_{v\in\mathcal{D}^*} \sum_{e:I(e)=v} \sum_{\beta\in C_3} P_\beta(e)\mathrm{d}z(e^*) + O(l\delta^\varepsilon)$$

We will show that the terms of the sum in this equation cancel locally for every $v \in \mathcal{D}^*$ by using identity (6.1), Lemma 6.1, a change of variables ($\gamma := \beta^{-1}\alpha$) in the summation over the elements of $C_3 \times C_3$, and finally equation (6.2):

$$\sum_{e:I(e)=v} \sum_{\beta\in C_3} P_\beta(e)\mathrm{d}z(e^*) = \sum_{\alpha,\beta\in C_3} P_\beta(\alpha(e_v))\mathrm{d}z(\alpha(e_v)^*) =$$

$$\sum_{\alpha,\beta\in C_3} P_{\beta^{-1}\beta}(\beta^{-1}\alpha(e_v))\mathrm{d}z(\alpha(e_v)^*) = \sum_{\alpha,\gamma\in C_3} P_1(\gamma(e_v))\mathrm{d}z(\alpha(e_v)^*) =$$

$$\sum_{\gamma\in C_3} P_1(\gamma(e_v)) \sum_{\alpha\in C_3} \mathrm{d}z(\alpha(e_v)^*) = \sum_{\gamma\in C_3} P_1(\gamma(e_v))\cdot 0 = 0$$

□

Note that we didn’t use any graph-theoretic property of the regular triangular lattice (apart from Proposition 5.1), because both Lemma 6.1 and this proof holds for any triangulation. The case turns out to be exactly the opposite if we want to prove that $h_1 + \tau h_\tau + \tau^2 h_{\tau^2}$ is analytic, because the ideas developed in the previous sections work only if the planar triangulation $\mathcal{G}$ has an embedding with equilateral triangular faces, thus $\mathcal{G}$ must be $\mathcal{L}_\Delta$. The proof is as follows:

If $P_\beta(e) = P_{\alpha\beta}(\alpha(e))$ and $P_\beta(e) = O(\delta^\varepsilon)$ is the only information we know about $P_\beta(e)$, then

$$\oint_\Gamma 1H_1(z) + \tau H_\tau(z) + \tau^2 H_{\tau^2}(z)\mathrm{d}z = \sum_{v\in\mathcal{D}^*}\sum_{e:I(e)=v}\sum_{\beta\in C_3}\beta P_\beta(e)\mathrm{d}z(e^*) + O(l\delta^\varepsilon),$$

and the only chance that the integral vanishes as $\delta \to 0$ is local cancellation for every vertex $v \in \mathcal{V}^*$.

$$\sum_{\alpha,\beta\in C_3}\beta P_\beta(\alpha(e_v))\mathrm{d}z(\alpha(e_v)^*) = \sum_{\alpha,\gamma\in C_3}\gamma^{-1}\alpha P_1(\gamma(e_v))\mathrm{d}z(\alpha(e_v)^*) \tag{6.6}$$

Since we do not know any relation between the values of $P_1(e_v)$, $P_1(\tau(e_v))$ and $P_1(\tau^2(e_v))$, we must have $\sum_\alpha \alpha\mathrm{d}z(\alpha(e_v)^*) = 0$. Together with equation (6.2), this implies that the vector $(\mathrm{d}z(1(e_v)^*), \mathrm{d}z(\tau(e_v)^*), \mathrm{d}z(\tau^2(e_v)^*))$ must be a complex constant times $(1,\tau,\tau^2)$, so the embedding of the lattice triangle v^* must be equilateral.

This is exactly the case of $\mathcal{L}_\Delta$:

$$\mathrm{d}z(\tau(e)^*) = \tau\mathrm{d}z(e^*)$$

holds for any oriented edge e of the dual hexagonal lattice. This completes the proof of Theorem 5.2, Theorem 4.1, and eventually Cardy's conjecture of conformal invariant crossing probabilities of critical percolation on the plane, in the case of Bernoulli site percolation on the regular triangular lattice.

7 Cardy's universal conjecture

We will discuss a more general form of Cardy's conjecture in this section, but this requires a careful treatment of embeddings: it is much harder to find the conformally invariant planar drawing of a lattice in the general case. To demonstrate the dependence of Cardy's conjecture on the embedding, we prove uniqueness (up to scaling and rotation) of the conformally invariant planar drawing.

Smirnov's groundbreaking theorem proved Cardy's conjecture only in a single special case, but an important part of the conjecture is *universality*: large-scale properties of critical percolation are supposed to be independent of the particular percolation model. There are many ways to generalize Bernoulli site percolation on $\mathcal{L}_\Delta$, but as the models become more general, the technical tools needed to mime the proof of the theorems presented in the previous sections become weaker.

First of all, even a proper formulation of Cardy's conjecture in its full generality needs a proper formulation of the planar embedding of the percolation model. It is convenient to use periodic graphs with periodic embedding.

Definition 7.1. *Let $\mathcal{A}$ be a finite index set. A periodic 2-dimensional lattice $\mathcal{L}$ is an infinite graph with vertex set*

$$\{\, v(a,m,n) \,:\, a \in \mathcal{A},\ m,n \in \mathbb{Z} \,\}$$

and the edge set $\mathcal{E}$ satisfies the following periodicity conditions for all $m,n \in \mathbb{Z}$:

$$\{v(a_1,m_1,n_1), v(a_2,m_2,n_2)\} \in \mathcal{E} \iff \{v(a_1,m_1+m,n_1+n), v(a_2,m_2+m,n_2+n)\} \in \mathcal{E}$$

Of course we assume that the lattice graph is connected and that the vertices are of finite degree.

If we want to define $\mathcal{L}_\Delta$ with these notations, then $\mathcal{A} = \{a\}$ is a one-element set, and edges of the lattice are of form

$$\{v(a,m,n), v(a,m+1,n)\},\ \{v(a,m,n), v(a,m,n+1)\},\ \text{or}\ \{v(a,m,n+1), v(a,m+1,n)\}.$$

We consider a periodic product measure on the red-green colouring of the vertices of $\mathcal{L}$:

$$\mathbf{P}(\, v(a,m,n) \text{ is green}\,) = p(a)$$

independently of the other vertices. $p(a) = \frac{1}{2}$ in Smirnov's model.

We will investigate the connectivity properties of the random subgraph spanned by the green vertices.

Definition 7.2. *A periodic embedding of $\mathcal{L}$ in the complex plane is defined by $z(a) \in \mathbb{C}$ for every $a \in \mathcal{A}$ and $\hat{z} \in \mathbb{C} \setminus \mathbb{R}$ in the following way:*

$$z(v(a,m,n)) = z(a) + m + n\hat{z}$$

The lattice of mesh δ is defined by the embedding $\delta \cdot z(v(a,m,n))$, but we will omit the explicit indication of δ.

We would like to talk about conformal invariance of the topological properties of the percolation process as the mesh of the lattice tends to zero, so the explicit value of $z(a)$ must be insignificant in the limit. A periodic embedding might be defined by two independent vectors: $z(v(a,1,0)) - z(v(a,0,0))$ and $z(v(a,0,1)) - z(v(a,0,0))$, but if we implicitly assume that the limit of the investigated probabilities is rotation and scale invariant then it is sufficient to formulate the conjecture for $z(v(a,1,0)) - z(v(a,0,0)) = 1$. Thus the embedding is essentially defined by the value of $\hat{z}$, and we will see later that a careful choice of this value is an important part of Cardy's conjecture. In fact, we have already seen that $\hat{z} = \tau$ is the right choice in the case of $\mathcal{L}_\Delta$.

Definitions 2.5, 2.6 and 4.1 can be generalized in a straightforward geometrical way, even if the lattice graph is not planar, although one might note that the definition of "counterclockwise orientation" is ambigous, because it can depend on the embedding of $\mathcal{L}$ in the plane. That is the reason why we assume that $\mathrm{Im}(\hat{z}) > 0$.

One more question remains: how do we define *critical percolation* in the case of inhomogenous percolation processes? An explicit way to do so is to assume that there is a uniform upper and lower bound on the horizontal and verical crossings of rectangles (similarly to Proposition 5.1):

Definition 7.3. *A 2-dimensional periodic Bernoulli site percolation process is critical if for every rectangle (Ω, a, b, c, d) with sides parallel to the real and imaginary axis there exists a $p > 0$ such that*

$$0 < p \leq \liminf_{\delta \to 0} \mathbf{P}(\mathsf{arc}(a,b) \overset{G}{\leftrightsquigarrow} \mathsf{arc}(c,d)) \leq$$
$$\limsup_{\delta \to 0} \mathbf{P}(\mathsf{arc}(a,b) \overset{G}{\leftrightsquigarrow} \mathsf{arc}(c,d)) \leq 1 - p < 1$$

Note that we did not mention the embedding of the graph explicitly, although the crossing probabilities can depend on the stretching of the embedding (the same way as they can depend on the stretchings of the rectangle (Ω, a, b, c, d)), but the fact that the crossing probabilities are bounded away from 0 and 1 is independent of the embedding.

Now we are able to formulate Cardy's conjecture in its universal form:

Conjecture 7.1. *For every critical 2-dimensional percolation process there exists a unique $\hat{z}$ with $\mathrm{Im}(\hat{z}) > 0$ such that the statement of Theorem 2.2 holds for a planar embedding of $\mathcal{L}$ with $\hat{z}$.*

If we believe in the validity of this conjecture, then it is natural that we formulate a more general conjecture corresponding to Theorem 4.1 as well. If we do so, then it is easy to see the reason why $\hat{z}$ is unique:

Claim. *If we assume that the embeddings with $\hat{z}_1$ and $\hat{z}_2$ both satisfy the general conjecture, then $\hat{z}_1 = \hat{z}_2$.*

Proof. Denote by Ψ the unique $\mathbb{R}$-linear transformation of the complex plane for which $\Psi(1) = 1$ and $\Psi(\hat{z}_1) = \hat{z}_2$ holds.

If $g(\hat{z}, z, \Delta, a(1)', a(\tau)', a(\tau^2)')$ denotes the function of Definition 4.2 then we must explicitly indicate dependence on $\hat{z}$, since the vertex set of the graph $\mathcal{G}$ is defined to be the "big" connected component (see Figure 2.2) of the subgraph of $\mathcal{L}$ spanned by the vertex set

$$\{v \in \mathcal{V}(\mathcal{L}) : z(v) \in \Delta\},$$

so $\mathcal{G}$, $\lim_{\delta\to 0} H_\beta^\delta = h_\beta$ and g depend on the value of $\hat{z}$.

If we use $\hat{z}_1$ for embedding $\mathcal{L}$, then the general conjecture and Lemma 4.1 tell us that

$$g(\hat{z}_1, z, \Delta, a(1)', a(\tau)', a(\tau^2)') = z$$

holds. If we apply Ψ to Δ and the embedding of $\mathcal{L}$ simultaneously, we get

$$g(\,\Psi(\hat{z}_1), \Psi(z), \Psi(\Delta), \Psi(a(1)'), \Psi(a(\tau)'), \Psi(a(\tau^2)')\,) = z,$$

because for each δ the underlying graph $\mathcal{G}$ remains the same (it is the intersection of the transformed lattice and the transformed domain), so the value of the limit of the functions $\sum_{\beta\in C_3} a(\beta)' H_\beta^\delta(v)$ remains the same as well. But the statements of the generalized conjecture hold for embedding with $\Psi(\hat{z}_1) = \hat{z}_2$ as well, so after substituting $w = \Psi(z)$ we get that

$$g(\hat{z}_2, w, \Psi(\Delta), \Psi(a(1)'), \Psi(a(\tau)'), \Psi(a(\tau^2)')) = \Psi^{-1}(w)$$

must be an analytic function of the variable w. So Ψ^{-1} and consequently Ψ must be a linear function in the complex sense as well: $\Psi(z) = cz$, but then $\Psi(1) = 1$ implies $c = 1$ and $\hat{z}_1 = \hat{z}_2$. $\square$

8 Conclusion

I described Cardy's conjecture about conformal invariance of crossing probabilities and presented a "streamlined" version of Smirnov's proof in my diploma thesis. His result isn't just an important cornerstone in the development of percolation theory (and the mathematical theory of statistical phisics), but the proof is a real gem: it combines distant branches of mathematics, relates the symmetries of the equilateral triangular lattice to those of harmonic conjugate triplets and finds the combinatorial core of the theory in the discrete $\frac{2\pi}{3}$-Cauchy-Riemann-equations. I believe that the technical simplifications introduced by this diploma thesis make these ideas even more clear.

In the presentation of the proof, I tried to put emhasis on the fact that we don't have to impose any smoothness condition on the Jordan curve $\partial\Omega$ if we want to show that $g : \Omega \to \Delta$ is a conformal mapping, although Smirnov points out in his paper that the solutions of the PDE presented in the end of Section 4 and the functions H_β^δ are both stable under perturbations of domains, so it is enough to consider domains with a smooth boundary. Nevertheless, we didn't have to use these approximation techniques, because the reformulation of Lemma 4.1 is powerful enough to show the uniqueness of h_β for an arbitrary Ω.

Smirnov's proof doesn't use technical estimates of percolation theory other than the consequence of the Russo-Seymour-Welsh theorem presented in Proposition 5.1, but as we pointed out in Subsection 6.2, this amount of information is enough to prove conformal invariance of critical percolation only on the simplest periodic triangulated lattice, $\mathcal{L}_\Delta$. In order to prove Cardy's conjecture for other lattices, we need more delicate estimates of critical exponents. Moreover, an important question is still unanswered: there are no conjectures (other than trivial symmetry arguments) about the true value of $\hat{z}$ (see Definition 7.2) for a general lattice.

References

[1] MILTON ABRAMOWITZ, IRENE A. STEGUN *Handbook of mathematical functions*. Dover, New York, 1970.

[2] GÁBOR HALÁSZ, *Bevezető komplex függvénytan*, egyetemi jegyzet, 1998.

[3] JOHN L. CARDY, Critical percolation in finite geometries. *J. Phys. A*, 25(4):L201-L206, 1992.

[4] GEOFFREY GRIMMETT, *Percolation*. Springer-Verlag, Berlin, second edition, 1999.

[5] HARRY KESTEN, *Percolation Theory for Mathematicians*. Birkhäuser, Boston, 1982.

[6] ROBERT LANGLANDS, PHILIPPE POULIOT, AND YVAN SAINT-AUBIN, Conformal invariance in two-dimensional percolation. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):1-61, 1994.

[7] MICHAEL REED, BARRY SIMON, *Methods of modern mathematical physics, Vol. I: Functional Analysis*. Academic Press, New York, 1980.

[8] STANISLAV SMIRNOV, Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C.R. Acad. Sci. Paris Sr. I Math.* 333 (2001), no. 3, 239-244.