1. Műveletek törtekkel, hatványokkal, gyökökkel.

(Válogatás a feladatgyűjteményből, és néhány hasznos tudnivaló.)

Mindig figyeljünk erre : $\sqrt{a^2} = |a| \quad (a \in \mathbf{R})$ $\Rightarrow$ Pl. : $\sqrt{(x^2-y^2)^6} = \sqrt{((x^2-y^2)^3)^2} = \left|(x^2-y^2)^3\right|$

$$\sqrt[2n]{a^{2n}} = |a| \ , \qquad \sqrt[2n+1]{a^{2n+1}} = a \qquad (a \in \mathbf{R},\ n \in \mathbf{N}^+) \qquad \text{!!!}$$

Hasznos azonosságok : $a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}\cdot b + \ldots + a\cdot b^{n-2} + b^{n-1}) \qquad (a,b \in \mathbf{R},\ n \in \mathbf{N}^+)$,

Pl. : $a = 1,\ \ b = q \neq 1 \quad \Rightarrow \quad 1 + q + q^2 + \ldots + q^{n-1} = \dfrac{1-q^n}{1-q}$ (ld. mértani sorozatok)

$n = 2$ esetén $\quad a^2 - b^2 = (a-b)\cdot(a+b)$

$n = 3 \quad a^3 - b^3 = (a-b)\cdot(a^2 + ab + b^2)$

$n = 4 \quad a^4 - b^4 = (a-b)\cdot(a^3 + a^2b + ab^2 + b^3)$

$$\Rightarrow \quad a^{2k} - b^{2k} = (a+b)\cdot(a^{2k-1} - a^{2k-2}\cdot b + a^{2k-3}\cdot b^2 - \ldots + a\cdot b^{2k-2} - b^{2k-1}) \qquad (a,b \in \mathbf{R},\ k \in \mathbf{N}^+) \ ,$$

$$a^{2k+1} + b^{2k+1} = (a+b)\cdot(a^{2k} - a^{2k-1}\cdot b + a^{2k-2}\cdot b^2 - \ldots + a\cdot b^{2k-1} - b^{2k}) \qquad (a,b \in \mathbf{R},\ k \in \mathbf{N}^+) \ .$$

Binomiális tétel :

$$\forall n \in \mathbf{N},\quad \forall x,y \in \mathbf{R}, \qquad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} \cdot y^k ,$$

ahol $\dbinom{n}{k} := \dfrac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k!} \quad (k \in \mathbf{N},\ 1 \le k \le n)$ és $\dbinom{n}{0} := 1$ (binomiális együtthatók) .

Pl.: $n = 2$ esetén $\quad (x+y)^2 = x^2 + 2xy + y^2$

$n = 3 \quad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$n = 4 \quad (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

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1. Segédeszköz használata nélkül számoljuk ki az alábbi kifejezések értékeit :

a.) $3\cdot[(-2)-(-3)] + (-2)\cdot(-3) = 3\cdot 1 + 6 = \boxed{9}$,

b.) $\dfrac{4\cdot\{[(2-3)\cdot 5 + 4]\cdot 2 + 3\} + 10}{7} = \dfrac{4\cdot\{[-1]\cdot 2 + 3\} + 10}{7} = \dfrac{4\cdot 1 + 10}{7} = \boxed{2}$,

c.) $[(\dfrac{3\cdot(7+2)-8}{-3} + 7)\cdot 2] : \dfrac{-6}{(-3)\cdot(-2)} = [(\dfrac{27-8}{-3} + 7)\cdot 2] : (-1) = (\dfrac{19}{3} - 7)\cdot 2 = \dfrac{19-21}{3}\cdot 2 = \boxed{-\dfrac{4}{3}}$,

d.) $6^{-3}\cdot(-2)^5\cdot 12^{-1}\cdot(-3)^4 = -6^{-3}\cdot 2^5\cdot 2^{-1}\cdot 6^{-1}\cdot 3^4 = -6^{-3}\cdot 6^4\cdot 6^{-1}\cdot = \boxed{-1}$,

e.) $\dfrac{26^{-4}\cdot 25^{-4}}{60^{-8}} + \left[(\dfrac{1}{1024})^{\frac{1}{5}} \right]^{-\frac{3}{2}} = (\dfrac{6\cdot 6\cdot 100}{26\cdot 25})^4 + (\dfrac{1}{2^{10}})^{-\frac{3}{10}} = (\dfrac{2^3\cdot 3^2}{13})^4 + 2^3 = \dfrac{2^{12}\cdot 3^8}{13^4} + 2^3 = \ldots$

f.) $(\dfrac{1}{6})^{-\frac{2}{3}} : (\dfrac{36}{125})^{-\frac{2}{3}} + (8^{-\frac{1}{3}})^{-2} = (\dfrac{6\cdot 36}{125})^{\frac{2}{3}} + 8^{\frac{2}{3}} = (\dfrac{6}{5})^2 + 2^2 = \dfrac{36}{25} + 4 = \boxed{\dfrac{136}{25}}$,

g.) $\left(\dfrac{12^4\cdot 5^5}{3^4} : \dfrac{2^7\cdot 55^6}{(-11)^6} \right)^{-2} = \left(4^4\cdot 5^5\cdot \dfrac{11^6}{2^7\cdot 5^6\cdot 11^6} \right)^{-2} = \left(2\cdot\dfrac{1}{5} \right)^{-2} = \boxed{\dfrac{25}{4}}$.

2. Írjuk fel prímhatványok szorzataként az alábbiakat :

a.) $24^2 \cdot 42^3 \cdot 12^2 \cdot 28 \cdot 18^3 = (2^3 \cdot 3)^2 \cdot (2 \cdot 3 \cdot 7)^3 \cdot (2^2 \cdot 3)^2 \cdot (2^2 \cdot 7) \cdot (2 \cdot 3^2)^3 =$

$= (2^6 \cdot 3^2) \cdot (2^3 \cdot 3^3 \cdot 7^3) \cdot (2^4 \cdot 3^2) \cdot (2^2 \cdot 7) \cdot (2^3 \cdot 3^6) = (2^6 \cdot 2^3 \cdot 2^4 \cdot 2^2 \cdot 2^3) \cdot (3^2 \cdot 3^3 \cdot 3^2 \cdot 3^6) \cdot (7^3 \cdot 7) =$

$\boxed{= 2^{18} \cdot 3^{13} \cdot 7^4}$,

b.) $\dfrac{3^5 \cdot 8^5 \cdot 20^4 \cdot 49}{16^4 \cdot 6^4 \cdot 70^2} = \dfrac{3^5 \cdot 2^{15} \cdot (2^8 \cdot 5^4) \cdot 7^2}{2^{16} \cdot (2^4 \cdot 3^4) \cdot (2^2 \cdot 5^2 \cdot 7^2)} = \dfrac{3^5 \cdot 2^{23} \cdot 5^4 \cdot 7^2}{2^{22} \cdot 3^4 \cdot 5^2 \cdot 7^2} = \boxed{= 2 \cdot 3 \cdot 5^2}$,

3. Segédeszköz használata nélkül számoljuk ki az alábbi kifejezések értékeit :

a.) $\dfrac{2^{10} + 2^{11} - 2^{12}}{2^9 + 2^{10}} = \dfrac{2^{10} \cdot (1 + 2 - 2^2)}{2^9 \cdot (1 + 2)} = \dfrac{2 \cdot (-1)}{3} = \boxed{-\dfrac{2}{3}}$,

b.) $\dfrac{1{,}6 \cdot 10^{-3} \cdot 2{,}5 \cdot 10^5}{2 \cdot 10^{-2}} = 0{,}8 \cdot 10^{-1} \cdot 2{,}5 \cdot 10^5 = \boxed{2 \cdot 10^4}$,

c.) $\dfrac{360000 \cdot 0{,}0000025}{0{,}009} = \dfrac{3{,}6 \cdot 10^5 \cdot 2{,}5 \cdot 10^{-6}}{9 \cdot 10^{-3}} = 4 \cdot 10^4 \cdot 2{,}5 \cdot 10^{-6} \cdot 10^3 = \boxed{10^2}$,

d.) $\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} + \sqrt{\dfrac{2+\sqrt{2}}{2-\sqrt{2}}} = \dfrac{(\sqrt{2-\sqrt{2}})^2 + (\sqrt{2+\sqrt{2}})^2}{\sqrt{2+\sqrt{2}} \cdot \sqrt{2-\sqrt{2}}} = \dfrac{(2-\sqrt{2}) + (2+\sqrt{2})}{\sqrt{4-2}} = \dfrac{4}{\sqrt{2}} \quad \boxed{= 2 \cdot \sqrt{2}}$,

e.) $(\sqrt{16 + 2 \cdot \sqrt{55}} - \sqrt{16 - \sqrt{220}})^2 = (\sqrt{(\sqrt{11} + \sqrt{5})^2} - \sqrt{(\sqrt{11} - \sqrt{5})^2})^2 = (\left|\sqrt{11} + \sqrt{5}\right| - \left|\sqrt{11} - \sqrt{5}\right|)^2 =$

$= ((\sqrt{11} + \sqrt{5}) - (\sqrt{11} - \sqrt{5}))^2 = (2 \cdot \sqrt{5})^2 \quad \boxed{= 20}$.

4. Rendezzük nagyság szerinti sorrendbe az alábbi számokat :

a.) $a = -10^3$, $b = \ln 5 < \ln 2.5^2 < \ln e^2 = 2$,

$c = \lg 100 = 2$, $d = -\dfrac{7}{8}$.

$\Rightarrow \quad \boxed{a < d < b < c}$.

b.) $a = 2^{\frac{2}{3}}$, $b = 4^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{4}}$, $c = 25^{\frac{1}{2}} = 5$,

$d = \dfrac{5}{100} = \dfrac{1}{20}$, $e = \sqrt{8} = 2^{\frac{3}{2}} = 2 \cdot \sqrt{2}$, $f = \sqrt[3]{27} = 3$,

b és d 1-nél kisebb pozitív számok, és nyilvánvaló, hogy $d < b$. A többi szám 1-nél nagyobb,

s mivel $2^{\frac{2}{3}} < 2^{\frac{3}{2}}$, és $\sqrt{2} < 1.5$ miatt $2 \cdot \sqrt{2} < 3$, a nagyság szerinti rendezés :

$\Rightarrow \quad \boxed{d < b < a < e < f < c}$.

c.) $a = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$, $b = \operatorname{tg} 45^\circ = 1$, $c = \cos 45^\circ = \dfrac{\sqrt{2}}{2}$,

$d = \operatorname{ctg}(-45^\circ) = -\operatorname{ctg}(45^\circ) = -1$, $e = \cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45^\circ = -\dfrac{\sqrt{2}}{2}$,

$\Rightarrow \boxed{d < e < c < a < b}$.

d.) $a = \sqrt{(-3)^2} = |-3| = 3$,

$b = \sin\dfrac{7\pi}{3} = \sin\dfrac{6\pi + \pi}{3} = \sin(2\pi + \dfrac{\pi}{3}) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$,

$c = \log_3 \dfrac{1}{9} = -2$,

$\Rightarrow \boxed{c < b < a}$.

e.) $a = \sqrt{7 + 4\cdot\sqrt{3}} = \sqrt{(2+\sqrt{3})^2} = |2+\sqrt{3}| = (2+\sqrt{3})$, $\in (3, 4)$

$b = \sqrt{11 - 6\cdot\sqrt{2}} = \sqrt{(3-\sqrt{2})^2} = |3-\sqrt{2}| = (3-\sqrt{2})$, $\in (1.5,\ 2)$

$c = \sqrt{9 - 4\cdot\sqrt{5}} = \sqrt{(2-\sqrt{5})^2} = |2-\sqrt{5}| = (\sqrt{5}-2)$, $\in (0,\ 0.5)$

$d = \sqrt{4 - 2\cdot\sqrt{3}} - \sqrt{4 + 2\cdot\sqrt{3}} = (\sqrt{(\sqrt{3}-1)^2} - \sqrt{(\sqrt{3}+1)^2} = |\sqrt{3}-1| - |\sqrt{3}+1| = (\sqrt{3}-1) - (\sqrt{3}+1) = -2$.

$\Rightarrow \boxed{d < c < b < a}$.

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5. Hozzuk egyszerűbb alakra az alábbi kifejezéseket :

a.) $(\sqrt{a^5 \cdot \sqrt[5]{a^2}}) \cdot (\sqrt{a^5} \cdot \sqrt[5]{a^2}) : \dfrac{\sqrt[4]{a}}{\sqrt[3]{a}}$ $(\ \Rightarrow\ a > 0\ )$ $= (\sqrt{a^5})^2 \cdot \sqrt[10]{a^2} \cdot \sqrt[5]{a^2} \cdot \dfrac{\sqrt[3]{a}}{\sqrt[4]{a}} =$

$= a^5 \cdot \sqrt[5]{a} \cdot \sqrt[5]{a^2} \cdot \dfrac{\sqrt[12]{a^4}}{\sqrt[12]{a^3}} = a^5 \cdot \sqrt[5]{a^3} \cdot \sqrt[12]{a} = a^5 \cdot \sqrt[60]{a^{36}} \cdot \sqrt[60]{a^5} = a^5 \cdot \sqrt[60]{a^{41}} = a^{5+\frac{41}{60}}$ $\boxed{= a^{\frac{341}{60}}}$,

b.) $\dfrac{(-2)^{n+1}\cdot(\frac{1}{2})^n}{(\sqrt[3]{64})^n + 16^{n/2}} = \dfrac{(-1)^{n+1}\cdot 2}{4^n + 4^n}$ $\boxed{= (-1)^{n+1}\cdot 4^{-n}}$,

c.) $\dfrac{9^{\frac{n}{2}+1} + (\sqrt{3})^{2n+2}}{36^{\frac{n}{2}} + (\sqrt{6})^n\cdot(\sqrt{3})^n\cdot(\sqrt{2})^n} = \dfrac{9\cdot 3^n + 3\cdot 3^n}{6^n + (\sqrt{6})^n\cdot(\sqrt{6})^n} = \dfrac{12\cdot 3^n}{2\cdot 2^n\cdot 3^n}$ $\boxed{= 6\cdot 2^{-n}}$,

d.) $\dfrac{16^{\frac{n}{2}} - 2^{2n+3}}{125^{\frac{n}{3}} + (\sqrt{5})^{2n+2}} = \dfrac{4^n - 8\cdot 4^n}{5^n + 5\cdot 5^n} = \dfrac{-7\cdot 4^n}{6\cdot 5^n}$ $\boxed{= -\frac{7}{6}\cdot 0.8^n}$,

e.) $\dfrac{a-b}{(a+b)^2}\cdot\sqrt{\dfrac{(a^2-b^2)^6}{(a-b)^{10}}} = \dfrac{a-b}{(a+b)^2}\cdot\sqrt{\dfrac{(a+b)^6}{(a-b)^4}} = \dfrac{a-b}{(a+b)^2}\cdot\dfrac{|(a+b)^3|}{(a-b)^2}$ $\boxed{= \dfrac{|a+b|}{a-b}}$ $(\ a \neq \pm b\ )$.

f.) $\dfrac{x^2-25}{x^2-3x}:\dfrac{x^2+5}{x^2-9} = \dfrac{(x-5)\cdot(x+5)}{x\cdot(x-3)}\cdot\dfrac{(x-3)\cdot(x+3)}{x^2+5} = \dfrac{(x^2-25)\cdot(x+3)}{x\cdot(x^2+5)} = \dfrac{x^3+3x^2-25x-75}{x^3+5x}$.

g.) $\dfrac{1-\dfrac{x^2}{x^2-1}}{2+\dfrac{3x-1}{1-x}} = \dfrac{\dfrac{x^2-1-x^2}{x^2-1}}{\dfrac{2-2x+3x-1}{1-x}} = \dfrac{1}{1-x^2}\cdot\dfrac{1-x}{1+x}$ $\boxed{=\dfrac{1}{(1+x)^2}}$ $(\ x\neq\pm1\ )$.

h.) $\dfrac{x-4}{x+4}-\dfrac{x+4}{x-4}+\dfrac{16x}{x^2-16} = \dfrac{(x-4)^2-(x+4)^2+16x}{x^2-16} = \dfrac{-8x-8x+16x}{x^2-16}$ $\boxed{=0}$ $(\ x\neq\pm4\ )$.

i.) $\left(\dfrac{2}{x^2-x}-\dfrac{2x}{1-x^2}\right)\cdot\dfrac{2x^2+2x}{x^3-1}+\dfrac{4}{x-1} = \left(\dfrac{2}{x\cdot(x-1)}+\dfrac{2x}{(x-1)\cdot(x+1)}\right)\cdot\dfrac{2x\cdot(x+1)}{(x-1)\cdot(x^2+x+1)}+\dfrac{4}{x-1} =$

$= \dfrac{2\cdot(x+1)+2x^2}{x\cdot(x-1)\cdot(x+1)}\cdot\dfrac{2x\cdot(x+1)}{(x-1)\cdot(x^2+x+1)}+\dfrac{4}{x-1} = \dfrac{2}{(x-1)}\cdot\dfrac{2}{(x-1)}+\dfrac{4}{x-1}$ $\boxed{=\dfrac{4x}{(x-1)^2}}$ $(\ x\neq\pm1,\ 0\ )$.

j.) $\left(\dfrac{2c}{c+2}-\dfrac{2c}{3c-6}+\dfrac{8c}{c^2-4}\right)\cdot\dfrac{c-2}{c^2-4c} = \left(\dfrac{1}{c+2}-\dfrac{1}{3\cdot(c-2)}+\dfrac{4}{(c-2)\cdot(c+2)}\right)\cdot 2c\cdot\dfrac{c-2}{c\cdot(c-4)} =$

$= \dfrac{3\cdot(c-2)-(c+2)+12}{3\cdot(c-2)\cdot(c+2)}\cdot 2\cdot\dfrac{c-2}{c-4} = \dfrac{2c+4}{3\cdot(c+2)}\cdot 2\cdot\dfrac{1}{c-4}$ $\boxed{=\dfrac{4}{3\cdot(c-4)}}$ $(\ c\neq\pm2,\ 0,\ 4\ )$.

k.) $\dfrac{\sqrt[4]{x\cdot\sqrt[3]{x}}}{\sqrt[6]{x}}$ $(\ \Rightarrow\ x>0\ )$ $= \dfrac{\sqrt[4]{\sqrt[3]{x^4}}}{\sqrt[6]{x}} = \dfrac{\sqrt[3]{\sqrt[4]{x^4}}}{\sqrt[6]{x}} = \dfrac{\sqrt[3]{x}}{\sqrt[6]{x}} = \dfrac{\sqrt[6]{x^2}}{\sqrt[6]{x}}$ $\boxed{=\sqrt[6]{x}}$.