SYMMETRIES OF HOROBALL PACKINGS RELATED TO FAMOUS 3-DIMENSIONAL HYPERBOLIC TILINGS

Robert Thijs Kozma(1) and Jenő Szirmai(2)

(2) Mathematician (Budapest, Hungary, 1964)

Address: (1) Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago IL 60607 USA and (1), (2) Budapest University of Technology and Economics, Institute of Mathematics, Department of Geometry, H-1521 Budapest, Hungary. E-mail: rkozma2@uic.edu, szirmai@math.bme.hu.

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Abstract. We discuss the symmetries of optimal packings in 3-dimensional hyperbolic space related to some famous polyhedral tilings, the Coxeter tilings with Schläfli symbols $(3, 3, 6)$ and $(4, 3, 6)$, as well as the Lambert-cube tilings with $(p, q) = (3, 6)$ and $(4, 4)$. We introduce the simplicial density function to reveal that the optimal ball packings of hyperbolic 3-space feature limiting objects called horoballs. We show that for both types of tilings there are multiple optimal packing arrangements with distinct symmetries and
various horoball types. Finally, we mention implications to and recent results in higher dimensional hyperbolic spaces.

**Keywords**: Coxeter tilings, hyperbolic geometry, horoball packings, Kleinian groups, Lambert-cube tilings, optimal packing density.

### 1. SYMMETRIES OF HYPERBOLIC SPACE

Hyperbolic geometry was discovered independently by J. Bolyai, Lobachevsky, and Gauss (unpublished) in the 19th century. The key property of hyperbolic space its constant negative curvature, as consequence of which the parallel postulate of Euclid does not hold. This fact gives rise to many interesting properties and questions where our Euclidean intuitions can easily fail us. Hyperbolic n-space \( H^n \) may be embedded into Euclidean n-space in various ways, in what follows we shall use models of \( H^n \) embedded into the unit n-sphere, specifically the Cayley-Klein ball model.

Images of the hyperbolic plane are abundant; one need not search long to find a myriad of hyperbolic plane kaleidoscopes. These resemble fractals on the disk with increasingly rich geometric patterns towards the boundary. Most readers are no doubt familiar with the work of M. C. Escher whose art made hyperbolic phenomena familiar to the general public. On the other hand there are many theoretical results on Kleinian groups, the symmetries hyperbolic 3-space \( H^3 \), but only very few illustrations. One of our highlights will be to describe the optimal ball packings of hyperbolic space using such symmetries.

The boundary of \( H^3 \) may be identified with the Riemann Sphere, the conformal automorphisms of which are given by Möbius transformations, see (Mumford et al., 2002) for detailed examples. The Möbius maps naturally extend from this boundary to isometries of the entire \( H^3 \). Kleinian groups are discrete subgroups of Möbius transformations, a special class of which are the Coxeter groups of \( H^3 \), most simply described as the group of reflections onto the sides of a hyperbolic polyhedron. In this
paper we will show how Coxeter groups and related symmetries give rise to optimal ball packing arrangements in hyperbolic space.

2. REGULAR BALL PACKINGS

In $n$-dimensional space of constant curvature, the locally optimal packing by balls of a fixed radius $r$ is achieved by arranging $n$ balls of equal radius $r$ at the $n+1$ vertices of a regular simplex with side lengths $2r$, so that the balls are tangent at the midpoints of each edge. In three dimensions, this is just four congruent balls centered at the vertices of a regular tetrahedron. This local density, called the *simplicial packing density*, is defined as the quotient of the volumes of the ball intersections with the simplex by the volume of the simplicial cell (Böröczky and Florian 1964, Kellerhals, 1998). It is not too difficult to show that the packing of the Euclidean plane by balls (i.e. by circular disks) arranged along a regular triangular lattice (or inscribed in its dual hexagonal lattice) gives the highest possible packing density. Unfortunately, for higher dimensional Euclidean spaces the regular simplex doesn’t tile. This is a significant complication. The Kepler conjecture from 1611 stated that in Euclidean 3-space the cannonball packing has the optimal packing with density of approximately 74%. The conjecture was proved by Thomas Hales only in 1998 using a state of the art computer-assisted proof.

On the other hand, in hyperbolic 3-space $H^3$ a result on the monotonicity of the simplicial density function states that as the radius $r$ of the four spheres in the simplicial packing increases, the simplicial packing density improves (Szirmai, 2012). If we consider $r \to \infty$, in the limit we obtain the totally asymptotic tetrahedron with vertices on the boundary of $H^3$ (the Riemann sphere). Coxeter showed that such asymptotic tetrahedra tile $H^3$, hence the optimally dense packing is realized in tiling, denoted by Schläfli symbol $(3, 3, 6)$. This shorthand denotes that the faces of a cell are triangles, three triangles come together at a vertex to form a cell, and six cells meet along each edge of the tiling. Böröczky and Florian (1964) later showed that this packing is indeed optimal, and has density slightly over 85%. Of course, in this extremal case the vertices of the tiling lie at infinity and the balls of the packing degenerate into limiting objects called...
horoballs. A **horosphere** in \( \mathbb{H}^3 \) is a hyperbolic 2-sphere with infinite radius that is centered at the boundary of the space. More formally, a horosphere is a 2-surface orthogonal to the set of parallel geodesics passing through a point of the absolute quadratic surface. A **horoball** is a horosphere together with its interior.

Any two horoballs are congruent in the classical sense; there exists a hyperbolic isometry mapping one to another. To distinguish horoballs of a packing we introduce the notion of horoball type with respect to a packing as in (Szirmai, 2005a, 2005b). One then can speak of a one-parameter family of concentric horoballs centered at each ideal point on the boundary of the model sphere. Concentric horoballs can then have a range of relative densities with respect to a fundamental domain of a regular tiling.

**Definition:** Two horoballs of a regular horoball packing are of the **same type** or **equipacked** if and only if their local packing densities with respect the fundamental domain are equal. Otherwise the horoballs are of **different type**.

By admitting horoballs of different types, in (Kozma et al., 2012) we gave several new examples of horoball packing arrangements based on totally asymptotic Coxeter tilings that yield the Böröczky-Florian packing density upper bound, showing that the optimal ball packing arrangement in \( \mathbb{H}^3 \) obtained from the simplicial density function is not unique. See Fig. 2 and 4 for the four packing arrangements. We found one additional packing in tiling \((3, 3, 6)\) by the asymptotic tetrahedron, and two in the cubic lattice \((4, 3, 6)\). Note that the term lattice used here not in the Euclidean sense. In the next sections we present some important regular lattices in \( \mathbb{H}^3 \) and describe some optimal lattice packings (regular packings) with balls centered at the vertices or each type of lattice.

### 3. COXETER TILINGS

A Coxeter tiling is a uniform tiling or tessellation of hyperbolic space by polyhedra (Coxeter, 1954). A Coxeter group \( \mathcal{G} \) is a finitely generated group defined by a presentation of the form \( \langle g_i | (g_i g_j)^{k_{ij}} = 1 \rangle \) where \( k_{ij} \) is a positive integer or \( \infty \) satisfying the assumptions \( k_{ii} = 1 \) and \( k_{ij} = k_{ji} \) (i.e. \( k_{ij} \) are elements of a Coxeter matrix). In our cases a Coxeter group \( \mathcal{G} \) acts by isometries of \( \mathbb{H}^3 \), in particular it acts by reflections on the sides
of a polyhedron where \( k_{ij} \) corresponds to the dihedral angle \( \frac{\pi}{k_{ij}} \) between sides whose reflection is given by \( g_i \) and \( g_j \).

There are four totally asymptotic Coxeter tilings in \( H^3 \), tilings by ideal regular tetrahedra, cubes, octahedra and dodecahedra. The optimal packings by horoballs of different types of each case were investigated in (Kozma et al., 2012). The methodology was to center a horoball at each lattice point of the tiling and vary the horoball types until the best packing density was obtained. We first discuss the optimal packings in the tetrahedral tiling (3, 3, 6) case motivated in Section 2, and then discuss the surprising cubic cases.

**Definition** The density \( \delta \) of the horoball packing of a Coxeter tiling \( T \) is defined as the quotient of the volumes of the all horoball intersections with the fundamental domain \( F(T) \) of the tiling with the volume of the fundamental domain:

\[
\delta = \frac{\sum_{i=1}^{n} \text{Vol}(B_i \cap F)}{\text{Vol}(F)}
\]

In the next two subsections we give a non-technical overview of the four optimal horoball packings of \( H^3 \). For details on the symmetry groups of the packings, and the methodology used to generate the figures we refer the reader to (Kozma et al., 2016).

### 3.1. Tetrahedral Tiling (3, 3, 6)

In this case the totally asymptotic hyperbolic tetrahedron is the fundamental domain of the packing. In the standard projective basis (for homogeneous coordinates) in the Cayley-Klein model the coordinates of the ideal vertices of the tetrahedron can be chosen as

\[
E_0 = (1, 0, 0, 1), \quad E_1 = \left( 1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right), \quad E_2 = \left( 1, \frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right), \quad E_3 = (1, 0, 1, 0).
\]

See Fig. 1. for a schematic diagram. There are two cases of horoball packings with balls centered at each vertex yielding the optimal packing density of approximately 0.85328. The horoball configuration in the fundamental domain extends to all of \( H^3 \) the Coxeter
symmetry group generated by reflections on the sides of the fundamental tetrahedron. Denote this arrangement as $B_{336}^1$.

3.1.1 Böröczky-Florian Case (1964)

This represents the equilibrium case where all horoballs are equipacked with respect to the fundamental domain (i.e., we have one horoball type). Horoballs meet along the midpoint of each edge in the model sphere. See Fig. 2(a), while noting the hyperbolic distortions in the Cayley-Klein model. The group of symmetries of the packing within in the fundamental domain coincides with the symmetries of the regular tetrahedron.

![Figure 1: Schematic diagram of fundamental domain of the (3, 3, 6) Coxeter tiling made up of one totally asymptotic tetrahedron, with one maximal horoball centered at $E$, tangent to the opposite facet.](image)

3.1.2 Kozma-Szirmai Case (2012)

This new case represents the extremal case where one horoball is the maximal permissible inside the fundamental domain in the sense that it is tangent to the face opposite its center in the tetrahedral cell (recall Fig. 1), and has largest possible relative density with respect to the fundamental domain. The remaining three horoballs are have smaller relative densities and are of the same type. The smaller horoballs are tangent to the large horoball along the edges of the fundamental domain. See Fig. 2(b). The symmetries of the packing
in the fundamental domain are the threefold rotational symmetries interchanging the small horoballs. Denote this arrangement as $B_{336}^2$.

Figure 2: Optimal horoball packings of the $(3, 3, 6)$ Coxeter tiling in the fundamental domain. The black simplex with vertices on the outer sphere is the totally asymptotic tetrahedron, and the four ellipsoids represent the four horoballs centered at the vertices of the simplex with metric distortions in the Cayley Klein model (outer sphere).
3.2. Cubic tiling (4, 3, 6)

In this case we have the totally asymptotic hyperbolic cube as the fundamental domain of the packing. In the standard projective basis of the Cayley-Klein model we set the coordinates of the cube to be

\[ E_0 = (1, 0, 0, 1), \quad E_1 = \left(1, -\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right), \quad E_2 = \left(1, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}, \frac{1}{3}\right), \]
\[ E_3 = \left(1, 0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right), \quad E_4 = \left(1, 0, 2\frac{\sqrt{2}}{3}, -\frac{1}{3}\right), \quad E_5 = \left(1, -\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, -\frac{1}{3}\right), \]
\[ E_6 = \left(1, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}\right), \quad E_7 = (1, 0, 0, -1). \]

See Fig. 3.

![Figure 3: Schematic diagram of the (4, 3, 6) totally asymptotic cubic tiling.](image)

Again, two optimal horoball packings configurations exist that yield the optimal packing density of approximately 0.85328, see Fig. 4. (a) and (b). The contact structures within the fundamental domain are distinct. In the first case there are two horoball types, the four with greater relative density are tangent at the midpoints of the facets, in the second one horoball of the largest admissible type is in the cubic cell. We refer to the first packing as the balanced case, and the second as the maximal case. Within the fundamental domain the balanced case exhibits threefold rotational symmetry about any interior diagonal of
the cubic cell, while the maximal case only exhibits threefold rotational symmetry about the axial diagonal passing through the center of the maximal horoball. Denote these arrangement as $B_{436}^1$ and $B_{436}^2$ respectively.

![Figure 4](image_url)

**Figure 4:** Optimal horoball packings of the (4, 3, 6) Coxeter tiling in the fundamental domain. The black cube is the totally asymptic fundamental domain, and the eight ellipsoids represent the eight horoballs centered at the vertices of the simplex with metric distortions in the Cayley Klein model (outer sphere). (a) Balanced case. (b) Maximal case.

### 3.3. Optimal Packing Density of $H^3$

The packing densities for the above mentioned configurations of the tetrahedral (3,3,6) and cubic (4,3,6) tiling agree with the Böröczky-Florian upper bound, hence we have the following result:

**Theorem 1** Four optimally dense horoball arrangements exist for $H^3$, two in the tetrahedral tiling with ball configurations $B_{i,336}^1$, $i=1,2$ and two in the cubic tiling (4,3,6)
with configurations $B_{436}^i$, $i=1, 2$, all four yielding the Böröczky-Florian backing density bound

$$\delta_{\text{opt}} \approx 0.85327609.$$
is maximal, i.e. \( \Gamma = \text{Aut}(T) \) the group of all bijections preserving the face incidence structure of \( T \), and the reflections on the facets of the cubes in \( T \) are in \( \Gamma \).

We classify such face transitive tilings, the (generalized) Lambert-cube tilings, as follows (see also Table 1):

I. \((p, q)\) with \( p > 2 \), and \( q = 2 \)
Classical Lambert tilings. The dihedral angles of the Lambert-cubes are \( \pi/p \) \((p > 2)\) at the three skew edges and \( \pi/2 \) at all other edges. Molnár described their metric realization in \( H^3 \) (Molnár, 1993), and Kellerhals found their volumes (Kellerhals, 1991).

II. \((p, q) = (4, 4), (3, 6)\)
These Lambert-cubes are realized in \( H^3 \), and can be divided up into hyperbolic simplices. They have ideal vertices that lie on the absolute quadric of \( H^3 \).

III. \((p, q) = (3, 3), (3, 4), (3, 5)\)
Additionally using the angle requirements the second author proved that these tilings are realized in \( H^3 \) (Szirmai, 1994).

In the remainder of this section we survey horoball packings of Lambert cube tilings with parameters \((p,q) = (3,6), (4,4)\). To this end introduce the standard 3-dimensional projective coordinate system \( P^3 \) with basis \( \{ b_i \} \) \((i = 0, 1, 2, 3)\), and assign coordinates to the vertices of Lambert-cubes with parameters \( c, d, x, y \) (see Fig. 6).

\[
\begin{align*}
E_0(1,0,0,0) & \sim e_0, \quad E_1(1,d,0,0) \sim e_1, \quad E_2(1,0,d,0) \sim e_2, \quad E_3(1,0,0,d) \sim e_3, \\
E_4(1,c,c,c) & \sim e_4, \quad E_5(1,0,x,y) \sim e_5, \quad E_6(1,y,0,x) \sim e_6, \quad E_7(1,x,y,0) \sim e_7.
\end{align*}
\]
Figure 6: Schematic diagram of Lambert cubes with vertices at infinity \((p,q) = (3,3), (3,4), (3,5), (3,6), (4,4), \text{ and } (p>2, q=2)\).

Figure 7: Schematic diagram of Lambert cubes with \((p,q) = (3,6), (4,4)\) and one horosphere centered at \(E_0\).

Six ideal vertices \(E_1, E_2, E_3, E_5, E_6, E_7\) lie on the boundary of \(H^3\), while \(E_0\) and \(E_4\) are proper vertices. We consider packings with horoballs centered at the ideal vertices. In (Szirmai, 1994 and 2005b) we proved the above described combinatorial Lambert cube tilings are indeed realized in hyperbolic space with the metric derived from the bilinear form \(\langle x, y \rangle = x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3\) where \(x = x^j b_j\) and \(y = x^j b_j\).

Coordinates \(c, d, x, y\) for the vertices of the lambert Lambert cubes were found for parameters \((p,q) = (3,3), (3,4), (3,5), (3,6), (4,4), \text{ and } (p>2, q=2)\). In the cases \(p>2, q\)
The parameter $d$ is given by $\cos \frac{p}{d^2} = \frac{d^2 + 1}{d^2}$. Their values are summarized in Table 1.

<table>
<thead>
<tr>
<th>$(p, q)$</th>
<th>$c$</th>
<th>$d$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 3)</td>
<td>0.52915026</td>
<td>0.88191710</td>
<td>0.66143783</td>
<td>0.66143783</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>0.53911695</td>
<td>0.94090399</td>
<td>0.61220809</td>
<td>0.75625274</td>
</tr>
<tr>
<td>(3, 5)</td>
<td>0.54261145</td>
<td>0.98159334</td>
<td>0.58048682</td>
<td>0.80221305</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>0.54427354</td>
<td>1</td>
<td>0.56032419</td>
<td>0.82827338</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>0.54091816</td>
<td>1</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$(p &gt; 2, 2)$</td>
<td>$c = \frac{d}{1+d^2}$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d(1 - d^2)$</td>
</tr>
</tbody>
</table>

In order to determine the horoball packing densities of the Lambert cube tilings we first need the volumes of Lambert cubes. In (Szirmai, 2003) the second author found the volumes of certain hyperbolic polyhedra by finding decomposition into orthoschemes with known volumes for Lambert-cubes $W_{pq}$ with parameters $(p,q) = (3,3), (3,4), (3,5), (3,6), (4,4)$. In the cases $p > 2, q = 2$ the volume $W_{pq}$ is obtained in (Kellerhals, 1991). The volumes for $(p,q) = (3,6), (4,4)$ are $Vol(W_{36}) = 3.15775682$ and $Vol(W_{44}) = 3.33769269$.

### 4.1. Horoball packings of Lambert-cube tilings with one horoball type

First we investigate horoball packings of the Lambert-cube tilings using a single horoball type at one ideal vertex of each cell. The symmetry group of the horoball is the reflection subgroup of $\Gamma$, with the Lambert cube as its fundamental domain. Without loss of generality, assume the horoball to be centered at $E_3(e_3)$. Clearly a requirement for the horoball to yield an optimal packing is that it be tangent to at least one facet of the Lambert cube not containing vertex $E_3(1,0,0,1)$ (see Fig. 7). Using the projective machinery, we can find the metric data of the optimal horoball type (Szirmai, 2005b). The intersection of the optimal horosphere and the Lambert cube is the horospherical region $A_{pq}$. The
volume of the intersection of the horoball with the Lambert cube is calculated with the classical formula of Bolyai, so the following definition of packing density is well defined.

**Definition:** The density of the horoball packing with one horoball type for the Lambert-cube is defined by the formula

\[
\rho_{pq} := \frac{1}{2} \frac{k \text{Area}(A_{pq})}{\text{Vol}(W_{pq})},
\]

where the metric constant of \( H^3 \) is set as \( k = \sqrt{\frac{1}{K}} = 1 \).

**Remark 1:** The intrinsic geometry of the horosphere is Euclidean, so the area of horospherical triangle \( A_{pq} \) is found using Heron’s formula.

**Remark 2:** For case \((p,q) = (4,4)\) the optimal horosphere (Fig. 7) contains the center of the Cayley-Klein model and is tangent to all three faces of the Lambert-cube that do not contain vertex \( E_5(1,0,0,1) \).

In Table 2 we summarize our results on the optimal horoball packings with one horoball type in the cases \((p, q) = (3, 6)\) and \((4, 4)\).

<table>
<thead>
<tr>
<th>((p,q))</th>
<th>(\text{Area}(A_{pq}))</th>
<th>(\frac{1}{2} \text{Area}(A_{pq}))</th>
<th>(pq)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3,6))</td>
<td>(\approx 1.80056665)</td>
<td>(\approx 0.90028333)</td>
<td>(\approx 0.28510217)</td>
</tr>
<tr>
<td>((4,4))</td>
<td>(\approx 2.91421356)</td>
<td>(\approx 1.45710678)</td>
<td>(\approx 0.43656110)</td>
</tr>
</tbody>
</table>

**Theorem 2** The densest horoball packing of the Lambert-cube tilings with \((p,q) = (3, 6)\) or \((4,4)\) with one horoball type at one vertex of the fundamental domain, where the
symmetry group of the packings is reflection subgroup of $\Gamma$ is achieved in case $(4, 4)$ with the maximal density $\delta \approx 0.43656110$.

4.2 The optimal horoball packing with six horoball types for the parameters $(p, q) = (3, 6), (4, 4)$

In the following we shall denote the horoball with center $E_i$ by $B_i$, $(i = 1, 2, 3, 5, 6, 7)$ (see Fig. 8). We allow six horoballs of different types (i.e. their local packing densities with respect to the Lambert cube may vary).

**Definition** The density of a horoball packing with six horoball types for the Lambert-cube is defined by the formula:

$$p^i := \frac{\text{Vol}(B_i) \cdot W_{pq}}{\text{Vol}(W_{pq})}, \quad i = 1, 2, 3, 5, 6, 7, \quad (p, q) = (3, 6), (4, 4).$$

![Figure 8: Six generic horoball types centered at each ideal vertex in a Lambert cube.](image)

![Figure 9: The $B^a_i$ ball arrangement, $(p, q) = (4, 4)$. $B^a_5$ is tangent to $B^a_3$ and $B^a_1$.](image)

We are interested in maximizing $p^i_{opt}$ for six horoball types in the Lambert-cube (see Fig. 9). We consider three main horoball packing arrangements $B^i_j$ ($i = a, b, c$) related to Lambert cube tilings as in (Szirmai, 2005b) defined as follows. For the first, let $B^a_2$
and $B_3^a$ denote two horoballs through point $M \ 1,0, \ 1 \ 2, 1 \ 2$. Ball $B_1^a$ is tangent to $B_3^a$ and $B_5^a$ by the 3-fold rotational symmetry of the Lambert-cube about axis $E_6E_4$. Let $B_6^a$ be the horoball with center at $E_6$ tangent to $B_3^a$. Then $B_5^a$ and $B_7^a$ are again defined using the same 3-fold symmetry, see Fig. 4, for the case of $(p,q) = (4,4)$. This packing arrangement is denoted by $\{B_i^a\}$.

If we blow up horoball $B_3^a$ and continuously vary the others while preserving the packing requirements and the tangency relations, then we arrive at a new horoball arrangement $\{B_i^b\}$. Denote the horoball passing through the point $E_6(1,0,0,0)$ with center $E_3$ by $B_3^b$. This was the "optimal horoball type" in Section 4.1. Introduce $B_1^b$ and $B_5^b$ tangent to $B_3^b$, as well as $B_7^b$, $B_6^b$ tangent to $B_3^b$, moreover horoball $B_7^b$ touches the horoballs $B_1^b$ and $B_5^b$.

Consider the the horoball set $\{B_i^b\}$ and expand the radius of $B_3^b$ preserving the packing requirements, until the horoball is no longer tangent to $B_3^b$. This horoball is
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In this case \( B_3^b = B_3^c = B_6^c \), and the horoballs \( B_1^c \), and \( B_2^c \) are tangent to \( B_7^c \). Thus we get the extremal horoball packing arrangement \( \{ B_7^c \} \).

In (Szirmai, 2005b) we proved the following two theorems.

**Theorem 3** The Lambert cube tiling \((p, q) = (4, 4)\) has three distinct optimal horoball packing arrangements \( P_a, P_b, \) and \( P_c \) with equal density of
\[
\frac{\frac{1}{\text{Vol}(B_i^j)} W_{44}}{0.76719471}, \quad i = 1,2,3,5,6,7, j = a,b,c,
\]
realized by horoball arrangements \( \{ B_i^a \}, \{ B_i^b \}, \) and \( \{ B_i^c \} \).

**Theorem 4** The optimal horoball packing arrangement of Lambert-cube tiling \((p, q) = (3, 6)\) is unique with horoball arrangement \( \{ B_i^a \} \) and has density
\[
\frac{\frac{1}{\text{Vol}(B_i^a)} W_{36}}{0.77012273}, \quad i = 1,2,3,5,6,7.
\]

**Remark:** If we only admit a single type of horoball in the packings of tilings \((p,q) = (3,6)\) and \((4,4)\) then we can only do significantly worse,
\[
\begin{align*}
\text{opt} &\quad 0.54248858, & \quad 0.34246947.
\end{align*}
\]

Again, there are multiple realizations of the same optimal packing densities, similar to the behavior we found for the Coxter tilings. These results emphasize that the idea of the one-parameter family of horoball types in a packing lead to new insights and better understanding of packing problems in n-dimensional hyperbolic space with \( n > 2 \).

**5. RESULTS IN HIGHER DIMENSIONS**

In (Kozma et al., 2015) we investigated ball packings in hyperbolic 4-space. Using the same techniques as in this paper we found several counterexamples to a conjecture of L. Fejes-Tóth regarding the upper bound of packing density in \( H^4 \) (Fejes Tóth, 1964). The
highest known packing density now is ~71.6%. The hyperbolic regular 24-cell and its
regular 4-dimensional honeycomb with Schlӓfli symbol (3,4,3,4) also yields this new
optimal packing density. Unfortunately for hyperbolic spaces with dimension n > 3, the
monotonicity of the simplicial density function is an open question, so we can only
conjecture the optimality of these new packings. Further results on horoball packings in higher dimensional hyperbolic spaces are given in (Szirmai, 2007, 2013).

REFERENCES


