cdlk Sztoch

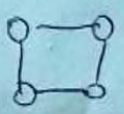
Ex 1 $p \mapsto \mathbb{P}_p[A]$ Any concrete graph has prob. $p^k(1-p)^{\binom{n}{2}-k}$ if it has k edges. The sum of these will be a polynomial of deg at most $\binom{n}{2}$.

If $p<p'$ Couple measures $\mathbb{P}_p$ and $\mathbb{P}_{p'}$ such that:
- $(U_e)_{e\in E}$ i.i.d. Unif$[0,1]$.
- Edge open iff $U_e \le p$.

Let W_p the set of open edges.
$\Rightarrow (W_p, W_{p'})$ is a coupling of $\mathbb{P}_p$ and $\mathbb{P}_{p'}$ s.t. $W_p \subseteq W_{p'}$ a.s.

We have: $\mathbb{P}_p[A] = \mathbb{P}[W_p \in A] \le \mathbb{P}[W_{p'} \in A] = \mathbb{P}_{p'}[A]$
(↓ A upward closed)

Ex 2 $A_n := \#\{C_4\text{s in } G(n,p)\}$.



4 vertices: $\binom{n}{4}$
4 edges: p^4

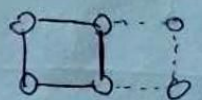
$\mathbb{E}[A_n] \asymp n^4p^4$ ← need $p \ge \frac{c}{n}$ to make this $\Omega(1)$.

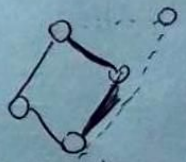
Hence $\mathbb{P}_p[A_n \ge 1] \le \mathbb{E}[A_n] \to 0$ if $p \ll \frac{1}{n}$.
(1)

Since A_n upward closed, $\exists$ threshold, and $\frac{1}{n}$ is a lower bd.

$$\mathrm{Var}[A_n] = \sum_{S,T\in\binom{[n]}{4}} \mathrm{Cov}[\mathbb{1}\{S\subseteq G\}, \mathbb{1}\{T\subseteq G\}] \quad (\le)$$

If $S\cap T=\emptyset \Rightarrow$ Cov $=0$

If $|S\cap T|=1 \Rightarrow$  $p^7\binom{n}{6} \asymp p^7n^6$

If $|S\cap T|=2 \Rightarrow$  $p^6\binom{n}{5} \asymp p^6n^5$

If $|S\cap T|=4 \Rightarrow p^4\binom{n}{4} \asymp p^4n^4$

$(\le) C\left(p^7n^6 + p^6n^5 + pn^4\right)$

If $p \gg \frac{1}{n}$: $\mathbb{P}[A_n = 0] \leq \frac{\mathrm{Var}[A_n]}{\mathbb{E}[A_n]^2} \leq \frac{C(p^7 n^6 + p^6 n^5 + p^4 n^4)}{(n^4 p^4)^2} =$ 2

$= C\left(\frac{1}{pn^2} + \frac{1}{p^2 n^3} + \frac{1}{p^4 n^4}\right) << C\left(\frac{1}{n} + \frac{1}{n} + 1\right)$

$\downarrow$

0

$\Rightarrow$ We have that $\asymp \frac{1}{n}$ is a threshold, i.e.

$\mathbb{P}_p[\#C_4 s > 0] = 1 - o(1)$ for $p \gg \frac{1}{n}$

$\mathbb{P}_p[\#C_4 s > 0] = o(1)$ for $p << \frac{1}{n}$

Ex 3 $p := \frac{\lambda \log n}{n}$ $\quad B_n := \#\{\text{isol. vertices in } G(n,p)\}$

$\mathbb{E}[B_n] = n \cdot \mathbb{P}[v \text{ is isolated}] = n \cdot (1-p)^{n-1} = n \cdot \left(1 - \frac{\lambda \log n}{n}\right)^{n-1}$

$\asymp n \cdot \exp(-\lambda \log n) = n^{1-\lambda} = \begin{cases} \to 0 & \lambda > 1 \\ \to \infty & \lambda < 1 \end{cases}$

$\Rightarrow \mathbb{P}[B_n \geq 1] \to 0$ if $\lambda > 1$ by Markov.

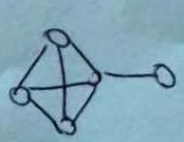
$\mathrm{Var}[B_n] = \sum_{v,w \in V(G)}' \mathrm{Cov}[\mathbb{1}\{v \text{ is isol}\}, \mathbb{1}\{w \text{ is isol}\}]$

$= n \cdot \mathrm{Var}[\mathbb{1}\{v \text{ is isol.}\}] + n(n-1)\left[\mathbb{P}[v \text{ and } w \text{ are isol}] - \mathbb{P}[v \text{ is isol}]^2\right] \leq n(1-p)^{n-1}\left(1 - (1-p)^{n-1}\right) +$

$+ n(n-1) \cdot \left[(1-p)^{2n-3} - (1-p)^{2n-2}\right] \asymp$

$\asymp n^{1-\lambda}(1 - n^{-\lambda}) + \underbrace{n^{2-2\lambda} - n^{2-2\lambda}}_{o(n^{2-2\lambda})} = n^{1-\lambda} + o(n^{2-2\lambda})$

$\Rightarrow \mathbb{P}[B_n = 0] \leq \frac{n^{1-\lambda} + o(n^{2-2\lambda})}{n^{2-2\lambda}} \to 0$ if $\lambda < 1$

Ex 4 H:  Let $\frac{5}{7} > \alpha > \frac{4}{6}$ $\quad p := n^{-\alpha}$.

$C_n := \#Hs$ in $G(n,p)$ $\quad \mathbb{E}[C_n] = \binom{n}{5} p^7 \asymp n^5 p^7 = o(1)$

if $p = o\left(\frac{1}{n^{5/7}}\right) \Leftrightarrow \alpha < \frac{5}{7}$

If $p << \frac{1}{n^{2/3}}$, then $\mathbb{P}[\#K_4 s > 0] \to 0 \Rightarrow \mathbb{P}[\#Hs > 0] \to 0$

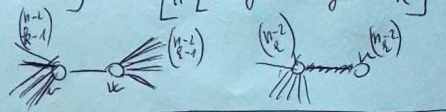
Ex 5 $X_k(n) := \#\{\text{degree } k \text{ vertices in } G(n,p)\}$ $p = \frac{\lambda}{n}$.

$$\mathbb{E}\left[\frac{X_k(n)}{n}\right] = \mathbb{E}\left[\mathbb{1}\{\text{vertex } v \text{ has degree } k\}\right] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$\to \mathbb{P}[\mathrm{Poi}(\lambda) = k] \qquad \mathbb{P}\left[\frac{X_\ell(n)}{n} - p_\lambda(\ell) \geq 1\right] \to 0$$

$$\mathrm{Var}[X_k(n)] = \mathrm{Var}\Big[\sum_{v,w \in V(G)} \mathrm{Cov}\left[\mathbb{1}\{\deg v = k\}, \mathbb{1}\{\deg w = k\}\right]$$

$$= n\left[\mathbb{P}[\deg v = k] - \mathbb{P}[\deg v = k]^2\right] + n(n-1)\left[\mathbb{P}[\deg v = \deg w = \ell, v \neq w] - \mathbb{P}[\deg w = \ell]^2\right] \leq n \cdot \mathbb{P}[\deg v = \ell] + n^2\left[\mathbb{P}[\deg v = \deg w = k] - \mathbb{P}[\deg v = k]^2\right]$$



$$\Rightarrow \mathrm{Var}[X_k(n)/n] \leq \frac{1}{n}\binom{n-1}{k} p^k (1-p)^{n-1-\ell} + \left[\binom{n-2}{k-1}^2 p^{2k-\ell}(1-p)^{2(n-k-1)} + \binom{n-2}{\ell}^2 p^{2\ell}(1-p)^{2(n-\ell-1)} - \binom{n-1}{\ell}^2 p^{2\ell}(1-p)^{2(n-1-\ell)}\right] \leq$$

I messed up here...

$$\leq \binom{n-2}{k-1}^2 p^{2\ell} + o(1) \leq \frac{n^{2(k-1)}}{p^{2\ell}} + o(1) = o(1)$$

$$\Rightarrow \mathbb{P}\left[\left|\frac{X_\ell(n)}{n} - p_\lambda(\ell)\right| > \varepsilon\right] \leq \mathbb{P}\left[\left|\frac{X_\ell(n)}{n} - \mathrm{Bin}(n,p,k)\right| > \varepsilon - |p_\lambda(\ell) - \mathrm{Bin}(n,p,\ell)|\right]$$

$\mathrm{Bin}(n,p,\ell) \to p_\lambda(\ell) - \varepsilon_1$ $\quad \frac{X_\ell(n)}{n} \to 1$ $\quad > \varepsilon$

$$\leq \frac{\mathrm{Var}\left[\frac{X_\ell(n)}{n}\right]}{(\varepsilon - o(1))^2} \to$$

Ex 6 $X \sim \text{Binom}(60, \frac{1}{2})$

$$\mathbb{P}[|X-30| \geq 20] = 2\,\mathbb{P}[X-30 \geq 20] = 2 \cdot \mathbb{P}[X \geq 50]$$

$$= 2 \cdot \mathbb{P}[e^{tX} \geq e^{50t}] \leq 2 \cdot \frac{\left(\frac{1}{2} + \frac{1}{2}e^t\right)^{60}}{e^{50t}} = \frac{1}{2}^{60} \cdot \left(\frac{(1+e^t)}{e^{5/6\,t}}\right)^{60}$$

Minimize: $f(t) = \log(1+e^t) - \frac{5}{6}t$ $\qquad f'(t) = \frac{e^t}{1+e^t} - \frac{5}{6} = 0$

$$\Rightarrow e^t = \frac{5}{6} + \frac{5}{6}e^t \Rightarrow e^t = 5 \Rightarrow t = \log 5$$

Taking $\inf_{t \in \mathbb{R}}$ we have: $2 \cdot \mathbb{P}(X \geq 50) \leq 2 \cdot 3^{60} \cdot 5^{-50}$.

Ex 7 $\mathbb{P}[|\text{Poi}(\lambda) - \lambda| > \delta\lambda] < C_\delta e^{-c_\delta \lambda}$

$$\mathbb{P}[|\text{Poi}(\lambda) - \lambda| > \delta\lambda] = \underbrace{\mathbb{P}[\text{Poi}(\lambda) - \lambda > \delta\lambda]}_{(*)} + \underbrace{\mathbb{P}[\text{Poi}(\lambda) - \lambda < -\delta\lambda]}_{(**)}$$

$$(*) = \mathbb{P}[\text{Poi}(\lambda) > (\delta+1)\lambda] = \mathbb{P}\left[e^{t \cdot \text{Poi}(\lambda)} > e^{t(\delta+1)\lambda}\right]$$

$$\leq \frac{e^{\lambda(e^t-1)}}{e^{t(\delta+1)\lambda}} = \exp\left[\lambda\left(\underbrace{e^t - 1 - t(\delta+1)}_{\text{minimize}}\right)\right]$$

$f(t) := e^t - 1 - t(\delta+1)$ $\qquad f'(t) = e^t - (\delta+1)$ $\qquad t = \log(\delta+1)$

$$\Rightarrow \inf_{t \in \mathbb{R}} \Rightarrow \mathbb{P}[\text{Poi}(\lambda) > (\delta+1)\lambda] \leq \exp\left[\lambda\left(\underbrace{\delta - \log(\delta+1)\cdot(\delta+1)}_{C_\delta^1}\right)\right]$$

$$(**)\ \mathbb{P}\left[e^{-t\,\text{Poi}(\lambda)} > e^{t(\delta-1)\lambda}\right] \leq \frac{e^{\lambda(e^{-t}-1)}}{e^{t(\delta-1)\lambda}} =$$

$$= \exp\left[\lambda\left(\underbrace{e^{-t} - 1 - (\delta-1)t}_{\text{minimize.}}\right)\right] \qquad f(t) = e^{-t} - 1 - (\delta-1)t$$

$$f'(t) = -e^{-t} - (\delta-1) = 0 \Rightarrow e^{-t} = 1-\delta \qquad t = \log\left(\frac{1}{1-\delta}\right)$$

$\Rightarrow \inf_{t\in\mathbb{R}} \Rightarrow \mathbb{P}[Poi(\lambda) < -\delta\lambda] \le \exp[\lambda(-\delta - (\delta-1)\log(\frac{1}{1-\delta})]$

$= \exp\left[\lambda\left(\underbrace{(1-\delta)\log\left(\frac{1}{1-\delta}\right)}_{\text{neg?}} - \delta\right)\right]$

$g(\delta) = (1-\delta)\log\left(\frac{1}{1-\delta}\right) - \delta \qquad g(0) = 0$

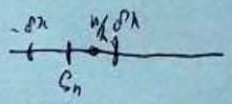
$g'(\delta) = -\log\left(\frac{1}{1-\delta}\right) + (1-\delta)(1-\delta)\cdot\frac{1}{(1-\delta)^2} - 1 = -\log\left(\frac{1}{1-\delta}\right) = \underset{0}{\log(1-\delta)}$

$\Leftrightarrow \delta = 0$

$g''(\delta) = \frac{-1}{1-\delta}$ < neg for $\delta < 1$ $\Rightarrow$ concave. $\Rightarrow$ unique max.

$\Rightarrow C_\delta^2 := (1-\delta)\log\left(\frac{1}{1-\delta}\right) - \delta$

$\hookrightarrow C_\delta := \max\{C_\delta^1, C_\delta^2\}$, $C_\delta := 2$



Ex 8 $\xi_i \sim Exp(\lambda)$ $\quad S_n = \sum_{i=1}^{n} \xi_i$ $\quad \mathbb{E}S_n = \frac{n}{\lambda}$

$\mathbb{P}\left[\left|S_n - \frac{n}{\lambda}\right| > \delta n\right] \le \underbrace{\mathbb{P}\left[S_n > \left(\delta + \frac{1}{\lambda}\right)n\right]}_{(*)} + \underbrace{\mathbb{P}\left[S_n < \left(-\delta + \frac{1}{\lambda}\right)n\right]}_{(**)} \Leftarrow$

S_n = time it took for Poisson(λ) P.P. to ring n times.

$\Leftrightarrow$ at time $\left(\delta + \frac{1}{\lambda}\right)n$, Poisson($\lambda$) rang less than n times.

$\Leftarrow \mathbb{P}\left[Poi\left[(1+\lambda\delta)n\right] < n\right] + \mathbb{P}\left[Poi(1-\lambda\delta) \ge n\right]$

$= \mathbb{P}\left[Poi\left((1+\lambda\delta)n\right) - (1+\lambda\delta)n < -\lambda\delta n\right] + \mathbb{P}\left[Poi\left((1-\lambda\delta)n\right) - (1-\lambda\delta)n \ge \lambda\delta n\right]$

~~$= \mathbb{P}\left[\left|Poi\left((1+\lambda\delta)n\right) - (\right.\right.$~~

$\eta_1 := \max\left\{\frac{\lambda\delta}{1+\lambda\delta}, \frac{\lambda\delta}{1-\lambda\delta}\right\}$ ~~$\eta_1 := \frac{\lambda\delta}{1+\lambda\delta}$~~

$\le 2\,\mathbb{P}\left[\left|Poi(\mu) - \mu\right| > \eta_1\mu\right] \le 2 \cdot C_\delta' \exp\left[-\hat{C}_\delta\mu\right]$

$= 2\,C_\delta \exp\left[-\underbrace{\hat{C}_\delta(1+\lambda\delta)}_{C_\delta} n\right]$

Or brute force:

$(*) = \mathbb{P}\left[e^{tS_n} > e^{tn(\delta+\frac{1}{\lambda})}\right] \le \frac{\frac{1}{(1-\frac{t}{\lambda})^\lambda}}{e^{tn(\delta+\frac{1}{\lambda})}}$

$\log \to -n\log\left(1-\frac{t}{\lambda}\right) \overset{\log}{\leq} nt\left(\delta+\frac{1}{\lambda}\right) = -n\underbrace{\left[\log\left(1-\frac{t}{\lambda}\right) + t\left(\delta+\frac{1}{\lambda}\right)\right]}_{f(t),\ \text{innere}}$

$$f'(t) = \frac{-1/\lambda}{1-t/\lambda} + \left(\delta+\frac{1}{\lambda}\right) = \frac{1}{-\lambda+t} + \left(\delta+\frac{1}{\lambda}\right) = 0$$

~~$\frac{1}{\lambda-t} = -\left(\delta+\frac{1}{\lambda}\right)$ $-\lambda+t = \frac{-1}{\delta+\frac{1}{\lambda}}$~~

~~$t = \lambda + \frac{1}{\delta+\frac{1}{\lambda}} = \frac{\lambda\delta+2}{\delta+1/\lambda}$~~

$$\frac{1}{t-\lambda} = -\delta-\frac{1}{\lambda}$$

$$t-\lambda = \frac{1}{-\delta-\frac{1}{\lambda}} = \frac{-\lambda}{\delta\lambda+1} \qquad \boxed{t = \frac{\delta\lambda^2}{\delta\lambda+1}}$$

$$C_\delta^1 = -\log(1+\delta\lambda) + \delta\lambda$$

(**) $\mathbb{P}\left[\exp(-tS_n) > \exp(-tn(1/\lambda-\delta))\right] \leq \frac{1}{(1+\frac{t}{\lambda})^n \exp(-nt(\frac{1}{\lambda}-\delta))} =$

~~$\log$~~ $\overset{-n}{\in} \exp\left[-n\underbrace{\left[\log(1+\frac{t}{\lambda}) - t(\frac{1}{\lambda}-\delta)\right]}_{f(t),\ \text{minimize}}\right]$

$$f'(t) = \frac{1/\lambda}{1+t/\lambda} - \left(\frac{1}{\lambda}-\delta\right) = 0$$

$$\frac{1}{\lambda+t} = \frac{1}{\lambda} - \delta \Rightarrow \lambda+t = \frac{\lambda}{1-\lambda\delta}$$

$$t = \frac{\lambda^2\delta}{1-\lambda\delta}$$

$\inf_{t>0}$: $\overset{*}{\leq} \exp\left[-n\left[\underbrace{-\log(1-\lambda\delta) - \lambda\delta}_{\text{pos?}}\right]\right]$

$$g(\delta) := -\lambda\delta + \log\left(\frac{1}{1-\lambda\delta}\right) \qquad g(0) = 0$$

$$g'(\delta) = -\lambda + (1-\lambda\delta)\frac{+\lambda}{(1-\lambda\delta)^2} = -\lambda + \frac{\lambda}{(1-\lambda\delta)^2} > 0$$

$\frac{1}{(1-\lambda\delta)^2} > 0$

$$\Rightarrow C_\delta^2 = \log\left(\frac{1}{1+\delta}\right) - \lambda\delta$$

$\to C_\delta^* := \max\{C_\delta^1, C_\delta^2\}$ $\qquad C_\delta := 2.$

Ex 9 $p, \alpha \in (0,1)$ $\alpha_n \to \alpha$ s.t. $\alpha_n n \in \mathbb{Z}\ \forall n$.

$$\mathbb{P}[\mathrm{Binom}(n,p) = \alpha_n n] = \binom{n}{\alpha_n n} p^{\alpha_n n} (1-p)^{(1-\alpha_n)n} = \frac{n!}{(\alpha_n n)!\,(n-\alpha_n n)!} p^{\alpha_n n} (1-p)^{(1-\alpha_n)n}$$

$$\sim \frac{n^{n+1/2} e^{-n} \sqrt{2\pi}}{(\alpha_n n)^{\alpha_n n + 1/2} e^{-\alpha_n n} \sqrt{2\pi}\, (n-\alpha_n n)^{(1-\alpha_n)n + 1/2} e^{-(n-\alpha_n n)} \sqrt{2\pi}} p^{\alpha_n n} (1-p)^{(1-\alpha_n)n}$$

$$-\log \mathbb{P}[\mathrm{Binom}(n,p) = \alpha_n n] = -(n+1/2)\log(n) + (\alpha_n n + \tfrac{1}{2})\log(\alpha_n n) + (n - \alpha_n n + \tfrac{1}{2})\log(n - \alpha_n n + \dots)$$

$$\Leftarrow \alpha_n n \log p \Leftarrow (1-\alpha_n) n \cdot \log(1-p) + o(1)$$

$$-\frac{\log \mathbb{P}[\mathrm{Binom}(n,p) = \alpha_n n]}{n} = -\log n + \alpha_n \log \alpha_n + \alpha_n \log n + (1-\alpha_n)\log n + (1-\alpha_n)\log(1-\alpha_n)$$

$$\Leftarrow \alpha_n \log p + (1-\alpha_n)\log(1-p) + o(1)$$

$$\longrightarrow \alpha \log \alpha + (1-\alpha)\log(1-\alpha) - \alpha \log p - (1-\alpha)\log(1-p)$$

$$= \alpha \log \frac{\alpha}{p} + (1-\alpha)\log\left(\frac{1-\alpha}{1-p}\right)$$

$$-\log \mathbb{P}[\mathrm{Binom}(n,p) = \lfloor np \rfloor] = -(n+\tfrac{1}{2})\log n + (\lfloor np \rfloor + \tfrac{1}{2})\log(\lfloor np \rfloor) + (n - \lfloor np \rfloor + \tfrac{1}{2}) \cdot$$

$$\cdot \log(n - \lfloor np \rfloor) + \lfloor np \rfloor \log p + (n - \lfloor np \rfloor)\log(1-p) + o(1)$$

$$\Leftarrow -n\log n + \tfrac{1}{2}\log n + np\log n + np\log p + n\log n - \tfrac{1}{2}\log n + \tfrac{1}{2}\log p$$

$$+ n\log n + n\log(1-p) - np\log n - np\log(1-p) + \tfrac{1}{2}\log n + \tfrac{1}{2}\log(1-p)$$

$$- np\log p - n\log(1-p) + np\log(1-p) + o(1) = \log n +$$

$$\frac{1}{2}\log p + \frac{1}{2}\log(1-p) + o(1)$$

$$\Longrightarrow \mathbb{P}[\mathrm{Binom}(n,p) = \lfloor pn \rfloor] \gtrsim \exp(-\log n) = \frac{1}{n}$$