Applications of Stochastics — Exercise sheet 1

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Notation. The probability measure for the Erdős-Rényi random graph $G(n,p)$ is denoted by $\mathbf{P}_p$.

Subsets of a base set S will be denoted by $\omega \in \{0,1\}^S$, thinking that $\omega(s) = 1$ iff $s \in \omega$.

The comparisons $\sim$, $\asymp$, $\ll$, $\gg$ are used as agreed in class.

"With high probability", abbreviated as "w.h.p.", means "with probability tending to 1".

$\triangleright$ **Exercise 1.** An event for the Erdős-Rényi random graph, $A \subset \{0,1\}^{\binom{n}{2}}$, is called *upward closed* or *increasing* if, whenever $\omega \in A$ and $\omega' \supseteq \omega$, then also $\omega' \in A$. Show that, for any such event A, other than the empty or the complete set, the function $p \mapsto \mathbf{P}_p[\,A\,]$ is a strictly increasing polynomial of degree at most $\binom{n}{2}$, with $\mathbf{P}_p[\,A\,] = p$ for $p \in \{0,1\}$. In particular, there exists a unique p such that $\mathbf{P}_p[\,A\,] = 1/2$; this value is usually called the *critical* (or *threshold*) *density*, and will be denoted by $p_c(n) = p_c^A(n)$.

$\triangleright$ **Exercise 2.** Find the order of magnitude of the critical density $p_c(n)$ for the random graph $G(n,p)$ containing a copy of the cycle C_4. (Hint: as in class, use the 1st and 2nd Moment Methods.)

The critical density for the connectedness of $G(n,p)$ is $p_c(n) = (1 + o(1))\frac{\ln n}{n}$, with a pretty sharp threshold. The following exercise is not a proof of this, just a small indication for the value.

$\triangleright$ **Exercise 3.** For $p = \frac{\lambda \ln n}{n}$, with $\lambda > 1$ fixed, show that, with probability tending to 1, there are no isolated vertices in $G(n,p)$. On the other hand, for $\lambda < 1$ fixed, there exist isolated vertices w.h.p.

The following is an example of subgraph containment where the Second Moment Method *fails*.

$\triangleright$ **Exercise 4.** Let H be the following graph with 5 vertices and 7 edges: a complete graph K_4 with an extra edge from one of the four vertices to a fifth vertex. Show that if $5/7 > \alpha > 4/6$, and $p = n^{-\alpha}$, then the expected number of copies of H in $G(n,p)$ goes to infinity, but nevertheless the probability that there is at least one copy goes to 0. What goes wrong with the 2nd Moment Method?

$\triangleright$ **Exercise 5.** Let $X_k(n)$ be the number of degree k vertices in the Erdős-Rényi random graph $G(n,\lambda/n)$, with any $\lambda \in \mathbb{R}_+$ fixed. Show that $X_k(n)/n$ converges in probability, as $n \to \infty$, to $\mathbf{P}[\,\mathsf{Poisson}(\lambda) = k\,]$. (Hint: the 1st moment of $X_k(n)$ is clear; then use the 2nd moment method.)

$\triangleright$ **Exercise 6.** Flip a fair coin 60 times, and let $X \sim \mathsf{Binom}(60, 1/2)$ be the number of heads. Using Markov's inequality for e^{tX} with the best possible t, which can be found by minimizing the convex function $f(t) = \log(1 + e^t) - \frac{5}{6}t$, show that

$$\mathbf{P}\big[\,|X - 30| \geq 20\,\big] \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}.$$

$\triangleright$ **Exercise 7.** Prove that for any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ such that

$$\mathbf{P}\big[\,|\mathsf{Poisson}(\lambda) - \lambda| > \delta\lambda\,\big] < C_\delta\, e^{-c_\delta \lambda},$$

for any $\lambda > 0$. (Hint: use the moment generating function of $\mathsf{Poisson}(\lambda)$.)

▷ **Exercise 8.** Let $\xi_i \sim \mathsf{Expon}(\lambda)$ i.i.d. random variables, and let $S_n := \xi_1 + \cdots + \xi_n$. Prove that for any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ (also depending on λ, of course) such that

$$\mathbf{P}\big[\, |S_n - \mathbf{E}S_n| > \delta n \,\big] < C_\delta \, e^{-c_\delta n}.$$

Hint: use the moment generating function of Expon or the previous Poisson exercise!

▷ **Exercise 9.** Let $p, \alpha \in (0,1)$ arbitrary, and let $\alpha_n \to \alpha$ such that $\alpha_n n \in \mathbb{Z}$ for every n. Using Stirling's formula, show that

$$\lim_{n\to\infty} \frac{-\log \mathbf{P}\big[\, \mathsf{Binom}(n,p) = \alpha_n n \,\big]}{n} = \alpha \log \frac{\alpha}{p} + (1-\alpha) \log \frac{1-\alpha}{1-p} \,.$$

When $\alpha = p$, we are getting that $\mathbf{P}\big[\, \mathsf{Binom}(n,p) = \alpha_n n \,\big]$ is only subexponentially small. In particular, roughly how large is $\mathbf{P}\big[\, \mathsf{Binom}(n,p) = \lfloor pn \rfloor \,\big]$?

The next bonus exercise contains some analytic details regarding the moment generating function. The main tool will be the *Dominated Convergence Theorem (DCT)*: if $\{X_n\}_{n\geq 1}$ and X and Y are random variables on the same probability space, with the almost sure pointwise convergence $\mathbf{P}[\, X_n \to X \,] = 1$, plus $|X_n| \leq Y$ holds almost surely for all n, where $\mathbf{E}Y < \infty$, then $\mathbf{E}|X_n - X| \to 0$, and thus $\mathbf{E}X_n \to \mathbf{E}X < \infty$.

▷ **Exercise 10.*** Assume that $m_X(t) := \mathbf{E}[\, e^{tX} \,] < \infty$ for some $t = t_0 > 0$, and let $\kappa_X(t) := \log m_X(t)$.

- **(a)** Show that $e^{tx} < 1 + e^{t_0 x}$ for all $0 \leq t \leq t_0$ and $x \in \mathbb{R}$. Deduce that $m_X(t) < \infty$ for all $0 \leq t \leq t_0$.
- **(b)** Using part (a) and the DCT, show that if $t_n \to t$, all of them in $[0, t_0]$, then $m_X(t_n) \to m_X(t)$. Thus $m_X(t)$ and $\kappa_X(t)$ are continuous functions of $t \in [0, t_0]$.
- **(c)** Show that $x < e^{tx}/t$ for any $t > 0$ and $x \in \mathbb{R}$. Deduce that $\mathbf{E}\big[\, X e^{tX} \,\big] < \infty$ if $0 < t \leq t_0/2$.
- **(d)** Using that $e^b - e^a = \int_a^b e^y \, dy$, show that $(e^{tx} - 1)/t \leq x e^{tx}$ for any $t > 0$ and $x \in \mathbb{R}$. Using part (c) and the DCT, show that $m'_X(0) = \mathbf{E}X < \infty$.
- **(e)** Deduce that $\kappa'_X(0) = \mathbf{E}X$. Deduce that if $\alpha > \mathbf{E}X$, then $\kappa_X(t) - \alpha t < 0$ for some $t \in (0, t_0)$.

The goal of the final bonus exercise is to present one way to pass from $G(n,p)$ to the $G(n,M)$ model.

▷ **Exercise 11.*** Fix $\delta > 0$ arbitrary, and let $p_n \in (0,1)$ and $M_n \in \big\{0, 1, \dots, \binom{n}{2}\big\}$ be two sequences satisfying $\binom{n}{2} p_n \to \infty$ and $(1+\delta)\binom{n}{2} p_n < M_n$ for all n. Let $A_n \subset \{0,1\}^{\binom{n}{2}}$ be a sequence of upward closed events such that $\mathbf{P}_{p_n}[\, A_n \,] \to 1$. Prove that

$$\mathbf{P}\big[\, G(n, M_n) \text{ satisfies } A_n \,\big] \to 1 \,, \qquad \text{as } n \to \infty \,.$$

In more detail:

- **(a)** Show that $\mathbf{P}\big[\, \mathsf{Binom}\big(\binom{n}{2}, p_n\big) < M_n \,\big] \to 1$.
- **(b)** Let $\mathcal{E}_n$ denote the number of edges in $G(n,p)$. Deduce from part (a) that $\mathbf{P}_{p_n}[\, A_n \mid \mathcal{E}_n < M_n \,] \to 1$.
- **(c)** Show that, for any $M \in \big\{0, 1, \dots, \binom{n}{2}\big\}$, we have $\mathbf{P}_{p_n}\big[\, A_n \,\big|\, \mathcal{E}_n = M \,\big] = \mathbf{P}\big[\, G(n,M) \text{ satisfies } A_n \,\big]$.
- **(d)** Deduce from part (c) that $\mathbf{P}_{p_n}\big[\, A_n \,\big|\, \mathcal{E}_n < M_n \,\big] \leq \mathbf{P}\big[\, G(n, M_n) \text{ satisfies } A_n \,\big]$.

Combining parts (b) and (d) concludes the exercise.