Inverse Eigenvalue Problems for Smooth Potential

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Abstract: We consider the inverse eigenvalue problem of the one-dimensional Schrödinger operator for finite intervals. We give sufficient conditions for finitely many partially known spectra and partial information on the potential to determine the Schrödinger operator on the whole interval.

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1 Introduction

Consider the Schrödinger operator

$$-y"(x) + q(x)y(x) = \lambda y(x) \quad \text{on} \quad (0, \pi) \tag{1.1}$$

with boundary conditions

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0 \tag{1.2}$$

$$y(\pi)\cos(\beta) + y'(\pi)\sin(\beta) = 0 \tag{1.3}$$

where $q(x) \in L_1(0, \pi)$. The sequence of eigenvalues (1.1)-(1.3) $\lambda_0 < \lambda_1 < \ldots$ form together the spectrum $\sigma(q, \alpha, \beta)$.

Our goal is to recover the potential q from a given set of eigenvalues (not necessarily taken from the same spectrum) and from partial knowledge of q. For more about this topic can be found in the paper of Korotyaev & Chelkak [**?**].

It is known that in most cases, two spectra is needed to recover the potential:

Theorem 1.1. *(Borg [3]) Let $q \in L_1(0, \pi)$, $\sigma_1 = \sigma_q(0, \beta)$, $\sigma_2 = \sigma_q(\alpha, \beta)$, $\sin\alpha \neq 0$ and $\sigma_2' = \sigma_2$, if $\sin\beta = 0$, $\sigma_2 \setminus \lambda_0$ else. Then $\sigma_1 \cup \sigma_2'$ determines the potential a.e. and no proper subset has the same property.*

Hochstadt and Lieberman observed that if the potential is known on half of the interval, then one spectrum is enough to determine the potential on the whole interval.

Theorem 1.2. *(Hochstadt & Lieberman [6]) If $q \in L_1(0, \pi)$ then q on $(0, \frac{\pi}{2})$ and the spectrum $\sigma(q, \alpha, \beta)$ determine q a.e. on $(0, \pi)$.*

This theorem has been further generalized by Gesztesy and Simon. The idea is that the knowledge of the eigenvalues can be replaced by information on the potential and its derivatives:

Theorem 1.3. *(Gesztesy & Simon [5]) Let $H = -\frac{d^2}{dx^2} + q$ in $L^2(0, 1)$ with boundary conditions (1.2), (1.3) and $\sin(\alpha) \neq 0$, and $\sin(\beta) \neq 0$. Suppose q is $C^{2k}(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ for some $k = 0, 1, \dots$ and for some $\varepsilon > 0$. Then q on $[0, \frac{1}{2}]$, α, and all eigenvalues of H except for $k + 1$ uniquely determine β and q on $(0, 1)$.*

The following can be said in the case when we do not have information about the potential on an interval, only in a single point:

Theorem 1.4. *Let $q_1, q_2 \in L_1(0, \pi)$ and assume that they are in $C^{2k}(0, \varepsilon)$ for some $0 < \varepsilon$, and $q_1(0) = q_2(0), \dots, q_1^{(2k)}(0) = q_2^{(2k)}(0)$. Furthermore suppose that for $\alpha_1 \neq \alpha_2$, $\sigma_{q_1}(\alpha_1, \beta_1) = \sigma_{q_2}(\alpha_1, \beta_2)$, and $\sigma_{q_1}(\alpha_2, \beta_1) = \sigma_{q_2}(\alpha_2, \beta_2)$ with $k + 1$ exceptions which are not known. Then $q_1(x) = q_2(x)$ a.e. on $(0, \pi)$ and $\beta_1 = \beta_2$.*

This theorem can be generalized by using information about q on $(0, a)$ for some $0 < a < \pi$, and the known eigenvalues can be derived from more than two (but finitely many) spectra. An almost optimal condition was given by Horváth.

Theorem 1.5. *(Horváth [8]) Let $1 \leq p < \infty$, $0 \leq a < \pi$, $q \in L_p(0, \pi)$ and let $\lambda_n \in \sigma(q, \alpha_n, \beta)$ be real numbers $\lambda_n \not\to -\infty$, $\sin(\beta) \neq 0$. If the set $\left\{e^{\pm 2i\sqrt{\lambda_n}x} : n \geq 1\right\}$ is closed in $L_p(a - \pi, \pi - a)$ then q on $(0, a)$ and the eigenvalues λ_n determine q in L_p. If $\sin(\beta) = 0$ then the eigenvalues λ_n, q on $(0, a)$ determine q in L_p if and only if the modified system $\left\{e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : n \geq 1\right\}$ with any $\mu \neq \lambda_n$ is closed in $L_p(a - \pi, \pi - a)$.*

One can test the closedness of this system using the Levinson-test.

Theorem 1.6. *(Levinson [10]) Let $0 \leq a \leq \pi$, $1 \leq p < \infty$, $1/p + 1/p' = 1$,* *let*

$$n(t) = \sum_{\lambda_n < t^2} 1$$

$N(t) = \int_1^\infty \frac{n(t)}{t} \mathrm{d}t$. *If*

$$\limsup_{r\to\infty} \left(N(r) - 2\left(1 - \frac{a}{\pi}\right) r + 1/p' \ln r \right) > -\infty \tag{1.4}$$

then the system $\{e^{i\sqrt{\lambda_n}x} : n \geq 1\}$ *is closed in* $L_p(a-\pi, \pi - a)$.

It is possible using the information on the derivatives of the potential to weaken (1.4), and the known eigenvalues still determine q.

Theorem 1.7. *Let* $1 \leq p < \infty$, $0 \leq a < \pi$, $q \in L_p(0,\pi) \cap C^{2k}(a-\varepsilon, a+\varepsilon)$ *if* $a \neq 0$, *and* $q \in L_p(0,\pi) \cap C^{2k}(0,\varepsilon)$ *if* $a = 0$ *for some* $\varepsilon > 0$.

Let $\sigma_j, j = 1, \ldots, N$ *be the spectrum of* (1.1)-(1.3) *with different* α_j *in* (1.2) *with the same* q *in* (1.1).

For each j *let* $S_j \subset \mathbb{N}$ *and suppose that* λ_n^j *is known for* $n \in S_j$. *Let*

$$n_j(t) = \sum_{\lambda_n^j < t^2, \quad \lambda_n^j \in S_j} 1,$$

$N_j(r) = \int_1^r \frac{n_j(t)}{t} \mathrm{d}t$, $1/p + 1/p' = 1$ *and suppose that there exists* $t_0 > 0$ *and* $c \in \mathbb{R}$ *such that for* $t \geq t_0$

$$\sum_{j=1}^N N_j(t) \geq 2\left(1 - \frac{a}{\pi}\right) t - \frac{1}{p'} \ln t - \tilde{k} \ln t - c, \tag{1.5}$$

where

$$\tilde{k} = \begin{cases} k-1 & \text{if } \sin(\beta_1), \sin(\beta_2) \neq 0 \quad p > 2 \\ k & \text{if } \sin(\beta_1), \sin(\beta_2) \neq 0 \quad p \leq 2 \\ k & \text{if } \sin(\beta_1) = \sin(\beta_2) = 0 \quad p > 2 \\ k+1 & \text{if } \sin(\beta_1) = \sin(\beta_2) = 0 \quad p \leq 2 \end{cases}$$

Then q *on* $(0,a)$, $q(a), q'(a), \ldots, q^{(2k)}(a)$ *and the known eigenvalues determine* q *in* $L_p(0,\pi)$ *and* β.

2 Proof of the theorems

Proof of Theorem 1.4. Let $u_1(\lambda, x)$ be the solution of (1.1) corresponding to q_1 for which $u_1(\lambda,\pi) = \sin(\beta_1)$ and $u_1'(\lambda,\pi) = -\cos(\beta_1)$ hold, and

$$m_1(\lambda) := \frac{u_1'(\lambda, 0)}{u_1(\lambda, 0)},$$

$m_2(\lambda)$ is defined similarly with q_2 and β_2. These are the so-called Weyl-Titchmarsh m-functions. It is known (Borg [4]) that q and $\tan(\beta)$ is determined by the m-function, so our goal is to show the equality of m_1 and m_2.

$$F(\lambda) := \frac{u_1'(\lambda,0)u_2(\lambda,0) - u_1(\lambda,0)u_2'(\lambda,0)}{\prod_{j=1}^{2}\prod_{\lambda_n^j \in \text{ both spectra}}\left(1 - \frac{\lambda}{\lambda_n^j}\right)}\prod_{i=1}^{k+1}\left(\lambda - \tilde{\lambda}_i\right) \tag{2.1}$$

$$= \frac{u_1'u_2 - u_1u_2'}{u_1u_2} \cdot \frac{u_1u_2}{\prod_{j=1}^{2}\prod_{\lambda_n^j \in \text{ both spectra}}\left(1 - \frac{\lambda}{\lambda_n^j}\right)} \cdot \prod_{i=1}^{k+1}\left(\lambda - \tilde{\lambda}_i\right) \cdot = I_1 \cdot I_2 \cdot I_3 \tag{2.2}$$

If $\lambda_n = 0$ then instead of $\left(1 - \frac{\lambda}{\lambda_n}\right)$ we write λ. All of the known eigenvalues in the denominator have a multiplicity of 1. They appear in the numerator as well, since $m_1(\lambda_{n,j}) = m_2(\lambda_{n,j})\,(j = 1, 2)$ because they satisfy the same boundary condition at 0. So $F(z)$ is an entire function.

On the other hand

$$I_1 = \frac{u_1'}{u_1} - \frac{u_2'}{u_2} = m_1 - m_2. \tag{2.3}$$

Our goal is to show that $F(z) \equiv 0$, from which it follows that $m_1 = m_2$ and then $q_1 = q_2$ and $\beta_1 = \beta_2$. We use a Phragmén–Lindelöf-type theorem:

Proposition 2.1. *(Simon & Gesztesi [5]) If $F(z)$ is an entire function,*

$$F(iy) \to 0 \tag{2.4}$$

as $|y| \to \infty$, $y \in \mathbb{R}$ and

$$\sup_{|z|=R_k} |F(z)| \le C_1 \mathrm{e}^{C_2 R_k^{\varrho}} \tag{2.5}$$

for some $0 \le \varrho < 1, 0 < C_1, C_2$ and some sequence $R_k \to \infty$ as $k \to \infty$ then $F(z) \equiv 0$.

First we prove that (2.4) holds. Denote

$$m(z, x) = \frac{u'(z, x)}{u(z, x)}$$

The m-function is known to have the following asymptotic expansion.

Proposition 2.2. *(Levitan [1]) If q is $C^{2k}(0, \delta)$ for some $\delta > 0$ and for some $k \in \mathbb{N}$ then*

$$m(z, x) = i\sqrt{z}\left(\sum_{l=0}^{2k+2} C_l(x) z^{-\frac{l}{2}} + o(z^{-k-1})\right), \qquad x \in [0, \delta] \tag{2.6}$$

as $|z| \to \infty$ *in any sector* $0 < \varepsilon < \arg(z) < \pi - \varepsilon$, *where* $C_0(x) = 1$, $C_1(x) = 0$, $C_2(x) = -\frac{1}{2}q(x)$, $C_j(x) = \frac{i}{2}C'_{j-1}(x) - \frac{1}{2}\sum_{l=1}^{j-1} C_l(x)C_{j-l}(x)$.

Using the equality of the derivatives of q_1 and q_2 we get that $C_l^{q_1}(0) - C_l^{q_2}(0) = 0, l = 0, \ldots, 2k+2$, so

$$I_1 = o\left(\lambda^{-k-\frac{1}{2}}\right), \tag{2.7}$$

as $|\lambda| \to \infty$ in a sector separated from the real axis.

To estimate the second factor we need the following propositions. By using the well-known eigenfunction asymptotics (see e.g. Levitan [2] Ch.I. or the proof of Theorem 1.8 in Horvaťh [7]), we get the following proposition.

Proposition 2.3. *We consider the Schrödinger equation*

$$-y"(x) + q(x)y(x) = zy(x) \quad \text{on} \quad (0, \pi)$$

for some $q \in L_1(0, \pi)$ *with the initial conditions* $y(\pi) = \sin(\gamma)$, $y'(\pi) = -\cos(\gamma)$. *Then for the solution* $v(x, z)$ *we have if* $\sin(\gamma) = 0$*:*

$$\begin{aligned} v(x,z) &= \frac{\sin(\sqrt{z}(\pi - x))}{\sqrt{z}} + O\left(\frac{e^{|\Im\sqrt{z}|(\pi-x)}}{|z|}\right) \\ v^{'}(x,z) &= -\cos(\sqrt{z}(\pi - x)) + O\left(\frac{e^{|\Im\sqrt{z}|(\pi-x)}}{\sqrt{|z|}}\right) \end{aligned} \tag{2.8}$$

if $\sin(\gamma) \neq 0$,

$$\begin{aligned} v(x,z) &= \sin(\gamma)\cos(\sqrt{z}(\pi - x)) + O\left(\frac{e^{|\Im\sqrt{z}|(\pi-x)}}{\sqrt{|z|}}\right) \\ v^{'}(x,z) &= \sin(\gamma)\sin(\sqrt{z}(\pi - x))\sqrt{z} + O\left(e^{|\Im\sqrt{z}|(\pi-x)}\right) \end{aligned} \tag{2.9}$$

where x *is fixed,* z *is large. The estimates are uniform in* $x \in [0, \pi]$.

Proposition 2.4. *(Zettl [11]) We consider the eigenvalues of the system* (1.1)-(1.3)*. If* $0 < \alpha, \beta < \pi$ *then*

$$\lambda_n = n^2 + \frac{2}{\pi}(\cot(\beta) - \cot(\alpha)) + \frac{1}{\pi}\int_0^\pi q + o(1), \tag{2.10}$$

where $n \geq 0$, $n \to \infty$.

For $\alpha = 0, 0 < \beta < \pi$

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + \frac{2}{\pi}\cot(\beta) + \frac{1}{\pi}\int_0^\pi q + o(1), \tag{2.11}$$

where $n \geq 0$, $n \to \infty$.

For $\alpha = 0, \beta = 0$

$$\lambda_n = n^2 + \frac{1}{\pi}\int_0^\pi q + o(1), \tag{2.12}$$

where $n \geq 1$, $n \to \infty$.

We examine the two factors of I_2 separately, with the notation $u_1(x,\lambda) = u(x,\lambda)$, $\beta_1 = \beta$. Following from Propositions 2.3 and 2.4, the accurate form of the factors depends on the boundary conditions. We have to distinguish four different cases:

(i) $\sin(\alpha) = 0$, $\sin(\beta) = 0$

$$\frac{u(x,\lambda)}{\prod\left(1 - \frac{\lambda}{\lambda_n}\right)} = \frac{u(x,\lambda)}{\prod_{n=1}^\infty\left(1 - \frac{\lambda}{n^2}\right)} \cdot \frac{\prod_{n=1}^\infty\left(1 - \frac{\lambda}{n^2}\right)}{\prod\left(1 - \frac{\lambda}{\lambda_n}\right)} \tag{2.13}$$

The denominator of the first factor can be calculated directly using the following identity.

$$\prod_{n=1}^\infty\left(1 - \frac{\lambda}{n^2}\right) = \frac{\sin(\pi\sqrt{\lambda})}{\pi\sqrt{\lambda}} \tag{2.14}$$

The second factor is bounded due to the following lemma:

Proposition 2.5. *(Horváth [7]) If* $z_n^* = z_n + O(1), z_n^* \neq 0$,

$$w(z) = \prod_{n=1}^\infty\left(1 - \frac{z}{z_n}\right), \qquad w^*(z) = \prod_{n=1}^\infty\left(1 - \frac{z}{z_n^*}\right) \tag{2.15}$$

then

$$\left|\frac{w(z)}{w^*(z)}\right|, \left|\frac{w^*(z)}{w(z)}\right| \tag{2.16}$$

are both bounded if $|z - z_n| > \delta, |z - z_n^*| > \delta \quad \forall n$.

If $|\lambda - \lambda_n| > \delta$, $|\lambda - n^2| > \delta$, using (2.8) we get

$$\frac{u(x,\lambda)}{\prod\left(1 - \frac{\lambda}{\lambda_n}\right)} = \frac{\frac{\sin(\pi\sqrt{\lambda})}{\sqrt{\lambda}} + O\left(\frac{e^{|\Im\pi\sqrt{\lambda}|}}{|\lambda|}\right)}{\frac{\sin(\pi\sqrt{\lambda})}{\sqrt{\lambda}}} = O(1) \tag{2.17}$$

We can choose δ small enough so that the excluded circles are disjoint for large n. From the maximum modulus principle it follows that (2.18) is valid on the entire complex plane.

If $\lambda_n = 0$ for some n, then in (2.13), the factor $1 - \frac{\lambda}{\lambda_n}$ is replaced by λ. We use similar calculations in the other three cases.

(ii) $\sin(\alpha) \neq 0$, $\sin(\beta) = 0$

$$\frac{u(x,\lambda)}{\prod\left(1-\frac{\lambda}{\lambda_n}\right)} = \frac{u(x,\lambda)}{\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\left(n+\frac{1}{2}\right)^2}\right)} \cdot \frac{\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\left(n+\frac{1}{2}\right)^2}\right)}{\prod\left(1-\frac{\lambda}{\lambda_n}\right)} \tag{2.18}$$

Using (2.8) and the identity

$$\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\left(n+\frac{1}{2}\right)^2}\right) = \cos(\pi\sqrt{\lambda}) \tag{2.19}$$

we get

$$= \frac{\frac{\sin(\pi\sqrt{\lambda})}{\sqrt{\lambda}} + O\left(\frac{e^{|\Im\pi\sqrt{\lambda}|}}{|\lambda|}\right)}{\cos(\pi\sqrt{\lambda})} = O\left(\frac{1}{\sqrt{\lambda}}\right) \tag{2.20}$$

(iii) $\sin(\alpha) = 0$, $\sin(\beta) \neq 0$

$$\frac{u(x,\lambda)}{\prod\left(1-\frac{\lambda}{\lambda_n}\right)} = \frac{u(x,\lambda)}{\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\left(n+\frac{1}{2}\right)^2}\right)} \cdot \frac{\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\left(n+\frac{1}{2}\right)^2}\right)}{\prod\left(1-\frac{\lambda}{\lambda_n}\right)} \tag{2.21}$$

Using (2.9) we get

$$= \frac{\cos(\pi\sqrt{\lambda}) + O\left(\frac{e^{|\Im\pi\sqrt{\lambda}|}}{\sqrt{|\lambda|}}\right)}{\cos(\pi\sqrt{\lambda})} = O(1) \tag{2.22}$$

(iv) $\sin(\alpha) \neq 0$, $\sin(\beta) \neq 0$

$$\frac{u(x,\lambda)}{\prod\left(1-\frac{\lambda}{\lambda_n}\right)} = \frac{u(x,\lambda)}{\prod_{n=1}^{\infty}\lambda\left(1-\frac{\lambda}{n^2}\right)} \cdot \frac{\prod_{n=1}^{\infty}\lambda\left(1-\frac{\lambda}{n^2}\right)}{\prod\left(1-\frac{\lambda}{\lambda_n}\right)} \tag{2.23}$$

$$= \frac{\cos(\pi\sqrt{\lambda}) + O\left(\frac{e^{|\Im\pi\sqrt{\lambda}|}}{\sqrt{|\lambda|}}\right)}{\sqrt{\lambda}\sin(\pi\sqrt{\lambda})} = O\left(\frac{1}{\sqrt{\lambda}}\right) \tag{2.24}$$

The same is true for $u_2(x,\lambda)$. Since $\alpha_1 \neq \alpha_2$, at least one of them is not equal to 0. It follows that

$$I_2 = O\left(\frac{1}{\sqrt{\lambda}}\right) \tag{2.25}$$

I_3 is clearly $O(\lambda^{k+1})$; putting (2.7) and (2.25) together we get

$$F(z) = o(\lambda^{-k-\frac{1}{2}})O\left(\frac{1}{\sqrt{\lambda}}\right)O(\lambda^{k+1}) = o(1) \tag{2.26}$$

Finally, to apply Proposition 2.1, we have to show that (2.5) holds.

We use Proposition 2.3 to estimate I_1. If $\sin(\beta_1) = 0$ and $\sin(\beta_2) = 0$, then from Proposition 2.3 we have:

$$I_1 = \frac{u_1^{'}u_2 - u_1u_2^{'}}{u_1u_2} = \frac{\frac{-\sin(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda})}{\sqrt{\lambda}} - \frac{-\sin(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda})}{\sqrt{\lambda}} + O\left(\frac{e^{2|\Im\sqrt{\lambda}\pi|}}{\lambda}\right)}{\frac{\sin(\pi\sqrt{\lambda})\sin(\pi\sqrt{\lambda})}{\lambda} + O\left(\frac{e^{2|\Im\sqrt{\lambda}\pi|}}{\lambda^{\frac{3}{2}}}\right)} = O(1) \tag{2.27}$$

If $\sin(\beta_1) \neq 0$ and $\sin(\beta_2) \neq 0$ then

$$I_1 = \frac{u_1^{'}u_2 - u_1u_2^{'}}{u_1u_2} = \frac{\sin(\beta_1)\sin(\beta_2)\sqrt{\lambda}\sin(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda})}{\sin(\beta_1)\sin(\beta_2)\cos(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda}) + O\left(\frac{e^{2|\Im\sqrt{\lambda}\pi|}}{\sqrt{\lambda}}\right)} - \frac{\sin(\beta_1)\sin(\beta_2)\sqrt{\lambda}\sin(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda}) + O\left(e^{2|\Im\sqrt{\lambda}\pi|}\right)}{\sin(\beta_1)\sin(\beta_2)\cos(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda}) + O\left(\frac{e^{2|\Im\sqrt{\lambda}\pi|}}{\sqrt{\lambda}}\right)} = O(1) \tag{2.28}$$

Finally, if $\sin(\beta_1) \neq 0$ and $\sin(\beta_2) = 0$ then

$$I_1 = \frac{u_1^{'}u_2 - u_1u_2^{'}}{u_1u_2} = \frac{\sin(\beta_1)\sqrt{\lambda}\sin(\pi\sqrt{\lambda})\frac{\sin(\pi\sqrt{\lambda})}{\sqrt{\lambda}} + \sin(\beta_1)\cos(\pi\sqrt{\lambda})\cos(\pi\sqrt{\lambda}) + O\left(e^{2|\Im\sqrt{\lambda}\pi|}\right)}{\sin(\beta_1)\cos(\pi\sqrt{\lambda})\frac{\sin(\pi\sqrt{\lambda})}{\sqrt{z}} + O\left(\frac{e^{2|\Im\sqrt{\lambda}\pi|}}{\sqrt{\lambda}}\right)} = O(\sqrt{z})$$

The estimates (2.27), (2.28), (2.29) of I_1 are valid on any circle on the complex plane which does not intersect the $\nu_1, \nu_2, \ldots$ roots of u_1u_2. Let R_k be a sequence of radii such that $|R_k - \nu_i| > \delta > 0, \forall k \forall i$, and $R_k \to \infty$ as $k \to \infty$. It is possible to find such R_k's since the roots of the solutions are separated.

So, irrespectively of the boundary conditions, $I_1 \cdot I_2$ can be estimated by a polynomial of λ.

Since I_3 is clearly $O(\lambda^{k+1})$, $F(z) \equiv 0$ follows from Proposition 2.1, which means $m_1 = m_2$ and then $q_1 = q_2$ and $\beta_1 = \beta_2$. The proof of Theorem 1.4 is complete.

To prove Theorem 1.7 we need the following Proposition.

Proposition 2.6. *Let us suppose that $\lambda_n \neq 0$ if $n \in S_j$. Define*

$$\omega_{S_j}(z) := \prod_{n \in S_j} \left(1 - \frac{z}{\lambda_n^j}\right) \tag{2.29}$$

In this case $\omega_{S_j}(z)$ is an entire function, and its roots are exactly λ_n^j. Furthermore

$$\ln|\omega_{S_j}(z)| = \int_1^\infty \frac{n_{S_j}(\sqrt{t})}{t} \frac{|z|^2 - xt}{|z|^2 - 2xt + t^2} \mathrm{d}t + O(x), \tag{2.30}$$

where $z = x + iy$.

Proof of Proposition 2.6.

$$\ln|\omega_{S_j}(z)|^2 = \sum_{\lambda_n^j \in S_j} \ln\left|1 - \frac{z}{\lambda_n^j}\right|^2 = \sum_{\lambda_n^j \in S_j} \ln\left(\left(1 - \frac{x}{\lambda_n}\right)^2 + \frac{y^2}{\lambda_n^{j\,2}}\right)$$

$$= \sum_{\lambda_n^j \in S_j} \ln\left(\left(1 - \frac{2x}{\lambda_n^j}\right)^2 + \frac{|z|^2}{\lambda_n^2}\right) = \int_1^\infty \ln\left(\left(1 - \frac{2x}{t}\right)^2 + \frac{|z|^2}{t^2}\right) \mathrm{d}n_{S_j}(\sqrt{t})$$

$$= \left[\ln\left(\left(1 - \frac{2x}{t}\right)^2 + \frac{|z|^2}{t^2}\right) n_{S_j}(\sqrt{t})\right]_1^\infty - \int_1^\infty n_{S_j}(\sqrt{t}) \frac{-\frac{2x}{t^2}\,\frac{2|z|^2}{t^3}}{1 - \frac{2x}{t} - \frac{|z|^2}{t^2}} \mathrm{d}t$$

$$= \int_1^\infty \frac{n_{S_j}(\sqrt{t})}{t} \frac{|z|^2 - xt}{|z|^2 - 2xt + t^2} \mathrm{d}t + O(x).$$

Proof of Theorem 1.7.

$$F(\lambda) = \frac{u_1'(\lambda,0)u_2(\lambda,0) - u_1(\lambda,0)u_2'(\lambda,0)}{\prod_{i=1}^{\tilde{k}}\left(\lambda - \tilde{\lambda}_i\right)\prod_{j=1}^N \prod_{n \in S_j}\left(1 - \frac{\lambda}{\lambda_n^j}\right)} \prod_{i=1}^{\tilde{k}}\left(\lambda - \tilde{\lambda}_i\right) \tag{2.31}$$

To estimate the denominator we need the integral representation from Proposition 2.6.

On the imaginary axis $x = 0$, using the substitution $t = \tau^2$ and integration by parts in (2.31), we obtain:

$$\left[N(\tau)\frac{2y^2}{y^2+\tau^4}\right]_1^\infty - \int_1^\infty N(\tau)\frac{-8y^2\tau^3}{(y^2+\tau^4)^2}\mathrm{d}t + O(1) \tag{2.32}$$

Since the first term is zero, from (1.5) we get

$$\begin{gathered}\ln\left|\prod_{j=1}^{N}\omega_j(iy)\right| \geq \int_1^\infty 2\left(1-\frac{a}{\pi}\right)\frac{-8y^2\tau^3}{(y^2+\tau^4)^2}\mathrm{d}\tau - \\ \int_1^\infty\left(\frac{1}{p'}+\tilde{k}\right)\ln\tau\frac{-8y^2\tau^3}{(y^2+\tau^4)^2}\mathrm{d}\tau - \int_1^\infty c\frac{-8y^2\tau^3}{(y^2+\tau^4)^2}\mathrm{d}\tau + O(1),\end{gathered} \tag{2.33}$$

Let $\tau = \sqrt{y}r$, then the first term becomes

$$2\left(1-\frac{a}{\pi}\right)\int_{\frac{1}{\sqrt{y}}}^\infty \frac{8r^4y^4\sqrt{y}}{y^4(1+r^4)^2}\mathrm{d}r = 2\left(1-\frac{a}{\pi}\right)\sqrt{y}\int_{\frac{1}{\sqrt{y}}}^\infty\frac{8r^4}{(1+r^4)}\mathrm{d}r \tag{2.34}$$

Substituting $\frac{1}{\sqrt{y}} = 0$ to the lower boundary:

$$2\left(1-\frac{a}{\pi}\right)\sqrt{y}\frac{\pi}{\sqrt{2}} \tag{2.35}$$

The second and the third term of (2.33) can be computed directly:

$$\begin{gathered}\int_1^\infty\left(\frac{1}{p'}+\tilde{k}\right)\ln\tau\frac{-8y^2\tau^3}{(y^2+\tau^4)^2}\mathrm{d}\tau + \int_1^\infty c\frac{-8y^2\tau^3}{(y^2+\tau^4)^2}\mathrm{d}\tau = \\ \left(\frac{1}{p'}+\tilde{k}\right)\frac{1}{2}\ln(1+y^2) + c\frac{2y^2}{y^2+1} = \left(\frac{1}{p'}+\tilde{k}\right)\ln y + O(1)\end{gathered} \tag{2.36}$$

If $\lambda_n = 0$ (which can happen at most once in each spectrum), then $\omega_{S_j}(z) := z\prod_{n\in S_j}\left(1-\frac{z}{\lambda_n}\right)$. The estimate of the denominator is still unchanged. The $+\ln(z)$ term is cancelled out because in (2.36) we have $\tilde{k}-1$ instead of $\tilde{k}$ since we counted this eigenvalue before. Depending on the boundary conditions we get:

(i) $\sin(\beta_1) \neq 0, \sin(\beta_2) \neq 0$

$$F(\lambda) = o(\lambda^{-k-\frac{1}{2}})\frac{O\left(e^{2|\Im\sqrt{\lambda}|(\pi-a)}\right)}{e^{2(\pi-a)\sqrt{\frac{\lambda}{2}}}}O\left(\lambda^{\tilde{k}+\frac{1}{p'}}\right) = o\left(\lambda^{-\frac{1}{2}+\frac{1}{p'}+\tilde{k}-k}\right) \tag{2.37}$$

(ii) $\sin(\beta_1) = 0, \sin\beta_2 = 0$

$$F(\lambda) = o(\lambda^{-k-\frac{1}{2}}) \frac{O\left(\frac{e^{2|\Im\sqrt{\lambda}|(\pi-a)}}{|\lambda|}\right)}{e^{2(\pi-a)\sqrt{\frac{\lambda}{2}}}} O\left(\lambda^{\tilde{k}+\frac{1}{p'}}\right) = o\left(\lambda^{-\frac{3}{2}+\frac{1}{p'}+\tilde{k}-k}\right) \tag{2.38}$$

Substituting into the definition of $\tilde{k}$, we get that $F(iy) \to 0$ as $|y| \to \infty$.

We use Proposition 2.3 to estimate $F(z)$. The same computation as in (2.27) yields $I_1 = 0(1)$, and the order of the numerator of I_2 is $\frac{1}{2}$

To calculate the order of the denominator of I_2, we use Borel's theorem (see Markushevich & Silverman [**?**] page 292).

Theorem 2.7. *(Borel) The order of the canonical product*

$$\Pi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \tag{2.39}$$

is equal to the convergence exponent of the sequence a_n*.*

Let n be large enough to $\lambda_n > 0$. From (1.5) we have

$$c_1\sqrt{\lambda_n} \leq N(\sqrt{\lambda_n}) = \int_1^{\sqrt{\lambda_n}} \frac{\sum_{\lambda_k < t^2} 1}{t} \mathrm{d}t \leq (n-1)\ln\sqrt{\lambda_n} \tag{2.40}$$

From $\ln x < x^{\frac{\varepsilon}{2}}, \forall\varepsilon > 0$ if x is large, it follows that

$$c_1\lambda_n^{\frac{1}{2}-\frac{\varepsilon}{2}} \leq n \qquad \forall\varepsilon > 0 \tag{2.41}$$

$$c_1\lambda_n \leq n^{\frac{2}{1-\varepsilon}} \qquad \forall\varepsilon > 0 \tag{2.42}$$

It follows that the convergence exponent of the known eigenvalues are at most $\frac{1}{2}$, so the order of the denominator is at least $\frac{1}{2}$, along any sequence of circles whose radii go to infinity and are not equal to any of the known eigenvalues. Such a sequence exists since there are finitely many spectra with separated elements.

Since I_3 is a polynomial of λ, the theorem follows from Proposition 2.1.

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