INVERSE EIGENVALUE PROBLEMS

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Abstract. In this article we consider inverse eigenvalue problems for the Schrödinger operator on a finite interval. We extend and strengthen previously known uniqueness theorems. A partially known potential is identified by some sets of eigenvalues and norming constants.

Key Words and Phrases: Inverse eigenvalue problem, Schrödinger operator on finite interval, norming constants, closed exponential systems

1. Introduction

We consider the Schrödinger operator on the finite interval $[0, \pi]$ defined by the equation

$$Ly = -y" + qy, \tag{1.1}$$

with the real-valued potential $q \in L_1([0, \pi])$.

The eigenvalue problem

$$\begin{aligned} Ly &= \lambda y \quad \text{on} \quad (0, \pi), && (1.2) \\ y(0) \cos \alpha + y'(0) \sin \alpha &= 0, && (1.3) \\ y(\pi) \cos \beta + y'(\pi) \sin \beta &= 0 && (1.4) \end{aligned}$$

defines the sequence of eigenvalues $\lambda_0 < \lambda_1 < \dots, \lambda_n \in \mathbb{R}$. Together they form the spectrum $\sigma(\alpha, \beta, q)$. Without loss of generality (by adding a sufficiently large constant to the potential) we may assume $0 \notin \sigma(\alpha, \beta, q)$ (which is assumed throughout the paper).

Let us fix λ and consider the initial value problems

$$\begin{aligned} Lu &= \lambda u \quad \text{on} \quad (0, \pi), && (1.5) \\ u(0) &= \sin \alpha, && (1.6) \\ u'(0) &= -\cos \alpha && (1.7) \end{aligned}$$

and

$$Lv = \lambda v \quad \text{on} \quad (0,\pi), \tag{1.8}$$
$$v(\pi) = \sin\beta, \tag{1.9}$$
$$v'(\pi) = -\cos\beta. \tag{1.10}$$

The solutions are denoted by $u(\lambda, x)$ and $v(\lambda, x)$ respectively. We define the norming constants $\tau(\lambda,\alpha,q)$ by $\tau(\lambda,\alpha,q) = \int_0^\pi |u(\lambda,x)|^2 dx$, and (for $\lambda \in \sigma(\alpha,\beta,q)$) $\kappa(\lambda,\beta,q)$ by $v(\lambda,x) = \kappa(\lambda,\beta,q)u(\lambda,x)$. Remark that for every λ, β, q there exists an α, unique $\mod \pi$, such that $\lambda \in \sigma(\alpha,\beta,q)$.

Our aim is to recover the potential q from four different types of given data:

(1) a set of eigenvalues possibly taken from infinitely many different spectra
(2) a set of norming constants belonging to known eigenvalues
(3) the potential itself on the interval $[0,a] \subset [0,\pi]$
(4) the smoothness of the potential in the neighbourhood of a.

The first result of this type was given by Ambarzumian in 1929:

Theorem 1.1. *(Ambarzumian, [?]) Let $q \in C([0,\pi])$ and $\sigma\left(\frac{\pi}{2},\frac{\pi}{2},q\right) = \{n^2, n \in \mathbb{N}\}$. Then $q \equiv 0$.*

We say that the set of eigenvalues determine q in $L_p(0,\pi)$ if there are no two different potentials $q, \tilde{q} \in L_p(0,\pi)$ which share all given eigenvalues. In 1946 Borg proved that in most cases two spectra are needed to recover the potential:

Theorem 1.2. *(Borg, [?]) Let $q \in L_1(0,\pi)$, $\sigma_1 = \sigma(0,\beta,q)$, $\sigma_2 = \sigma(\alpha,\beta,q)$, $\sin\alpha \neq 0$ and $\sigma_2' = \sigma_2$, if $\sin\beta = 0$, and $\sigma_2' = \sigma_2 \setminus \lambda_0$ if $\sin\beta \neq 0$. Then $\sigma_1 \cup \sigma_2'$ determines the potential a.e. and no proper subset has the same property.*

Hochstadt and Lieberman discovered in 1978 that if the potential is known on half of the interval, then one spectrum is enough to determine the potential on the whole interval:

Theorem 1.3. *(Hochstadt & Lieberman [?]) If $q \in L_1(0,\pi)$ then q on $\left(0, \frac{\pi}{2}\right)$ and the spectrum $\sigma(\alpha,\beta,q)$ determine q a.e. on $[0,\pi]$.*

This theorem has been further generalised by Gesztesy and Simon. They observed that the knowledge of the eigenvalues can be replaced by information on the potential and its derivatives:

Theorem 1.4. *(Gesztesy & Simon [?]) Let $H = -\frac{d^2}{dx^2} + q$ in $L^2(0,\pi)$ with boundary conditions (1.3), (1.4) and $\sin\alpha \neq 0$, $\sin\beta \neq 0$. Suppose q is $C^{2k}\left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right)$ for some $k \in \mathbb{N}$ and for some $\varepsilon > 0$. Then q on $\left[0, \frac{\pi}{2}\right]$, α and all eigenvalues of H except for $k+1$ uniquely determine $\tan\beta$ and q on $(0,\pi)$.*

For any real sequence $S = \{\mu_n | n \geq 0\}$, $\mu_n \to \infty$ define the counting function

$$n_S(t) = \sum_{\mu_n \leq t} 1.$$

Another result from the same paper is

Theorem 1.5. *(Gesztesy & Simon [?]) Let $q \in L_1(0,\pi)$, $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. If $S \subset \sigma = \sigma(\alpha,\beta,q)$ satisfies*

$$n_S(t) \geq 2\left(1 - \frac{a}{\pi}\right) n_\sigma(t) + \frac{a}{\pi} - \frac{1}{2} \text{ for large } t \tag{1.11}$$

then q on $(0,a)$, α and S uniquely determine q a.e. and $\tan\beta$.

We mention the following similar results using extra information on the smoothness of q:

Theorem 1.6. *(L. Amour et al [?], Theorem 1.1.) Let $\sin\alpha \neq 0$, $\sin\beta_i \neq 0$, $i = 1,2$, $\pi/2 \leq a < \pi$ and $p \in [1,\infty)$. Suppose that q_1, $q_2 \in L_1(0,\pi)$, $q_1 = q_2$ a.e. on $[0,a]$ and $q_1 - q_2 \in L_p([a,\pi])$. Consider an infinite set*

$$S \subset \sigma(\alpha,\beta_1,q_1) \cap \sigma(\alpha,\beta_2,q_2).$$

Assume that there exists a real number C such that

(1.12)
$$2\left(1-\frac{a}{\pi}\right)n_\sigma(t)+C\ge n_S(t)\ge 2\left(1-\frac{a}{\pi}\right)n_\sigma(t)+\frac{1}{2p}+2\frac{a}{\pi}-2$$

for $t\in S$ large enough, where σ denotes either of $\sigma(\alpha,\beta_i,q_i)$. Then $\tan\beta_1=\tan\beta_2$ and $q_1=q_2$ a.e.

Theorem 1.7. *(L. Amour et al [?], Theorem 1.1) Let $k\in\{0,1,2\}$, q_1, $q_2\in W^{k,1}([0,\pi])$ and $\sin\alpha\ne 0$, $\sin\beta_1\ne 0$, $\sin\beta_2\ne 0$. Let $S\subset\sigma(\alpha,\beta_1,q_1)\cap\sigma(\alpha,\beta_2,q_2)$. Fix $a\in[\frac{\pi}{2},\pi)$ and $1\le p\le\infty$. Suppose that $q_1=q_2$ on $[0,a]$ and $q_1-q_2\in W^{k,p}([a,\pi])$. Assume that*

$$n_s(t)\ge 2\left(1-\frac{a}{\pi}\right)n_{\sigma(\alpha,\beta_1,q_1)}(t)-\frac{k}{2}+\frac{1}{2p}-\frac{1}{2}-\left(1-\frac{a}{\pi}\right), \tag{1.13}$$

for $t\in\sigma(\alpha,\beta_1,q_1)$, t large enough. Then $\tan\beta_1=\tan\beta_2$ and $q_1=q_2$ a.e.

Theorem 1.8. *(L. Amour et al [?] 1.2) In the above theorem we can replace condition (1.13) by*

$$2\left(1-\frac{a}{\pi}\right)n_{\sigma(\alpha,\beta_1,q_1)}(t)+C\ge n_s(t)\ge \tag{1.14}$$
$$2\left(1-\frac{a}{\pi}\right)n_{\sigma(\alpha,\beta_1,q_1)}(t)-\frac{k}{2}+\frac{1}{2p}-2\left(1-\frac{a}{\pi}\right),$$

$t\in S$, t large enough.

In 2012 Wei and Xu showed that the knowledge of the eigenvalues can be replaced by the knowledge of norming constants. They considered the constants

$$k_w(\lambda,\beta,q)=\begin{cases}\frac{v(\lambda,0)}{\sin\beta}=\frac{\kappa(\lambda,\beta,q)\sin\alpha}{\sin\beta} \text{ if } \sin\alpha\ne 0,\ \sin\beta\ne 0,\\ \frac{v'(\lambda,0)}{\sin\beta}=\frac{\kappa(\lambda,\beta,q)}{\sin\beta} \text{ if } \sin\alpha=0,\ \sin\beta\ne 0,\\ v(\lambda,0)=\kappa(\lambda,\beta,q)\sin\alpha \text{ if } \sin\alpha\ne 0,\ \sin\beta=0,\\ v'(\lambda,0)=\kappa(\lambda,\beta,q) \text{ if } \sin\alpha=0,\ \sin\beta=0.\end{cases}.$$

Theorem 1.9. *(Wei & Xu [?]) Let $\sin\alpha\ne 0$, $\sin\beta_1\ne 0$, $\sin\beta_2\ne 0$, $k\in\mathbb{N}$, $\varepsilon>0$ and q_1, $q_2\in C^{2k-1}[0,\varepsilon)$. Assume that $\sigma(\alpha,\beta_1,q_1)=\sigma(\alpha,\beta_2,q_2)$, $q_1^{(j)}(0)=q_2^{(j)}(0)$ for $j=0,1,\dots,2k-1$*

and finally that $k_w(\lambda, \beta_1, q_1) = k_w(\lambda, \beta_2, q_2)$ *holds for all eigenvalues* $\lambda \in \sigma(\alpha, \beta_1, q_1)$ *with* $k+1$ *exceptions at most. Then* $\tan\beta_1 = \tan\beta_2$ *and* $q_1 = q_2$ *a.e.*

Theorem 1.10. *(Wei & Xu [?]) Let* $a \in [0, \pi/2)$, $\sin\alpha \neq 0$, $\sin\beta_1 \neq 0$, $\sin\beta_2 \neq 0$. *Assume that* $q_1 = q_2$ *a.e. on* $[0, a]$, q_1, $q_2 \in C^n(a-\varepsilon, a+\varepsilon)$ *for some* $n \in \mathbb{N}_0$ *and* $\varepsilon > 0$. *Assume that* $\sigma(\alpha, \beta_1, q_1) = \sigma(\alpha, \beta_2, q_2)$ *Let us suppose that for an infinite set* $S \subset \sigma(\alpha, \beta_1, q_1)$, $\kappa_w(\lambda, \beta_1, q_1) = \kappa_w(\lambda, \beta_2, q_2)$ *if* $\lambda \in S$. *Furthermore assume that*

$$n_S(t) \geq \left(1 - 2\frac{a}{\pi}\right) n_{\sigma(\alpha,\beta_1,q_1)}(t) + \frac{a}{\pi} - \frac{n+3}{2}$$

for all sufficiently large $t \in \mathbb{R}$. *Then* $\tan\beta_1 = \tan\beta_2$ *and* $q_1 = q_2$ *a.e.*
Similar statements hold if we write

$$n_S(t) \geq \left(1 - 2\frac{a}{\pi}\right) n_{\sigma(\alpha,\beta_1,q_1)}(t) - \frac{a}{\pi} - \frac{n+1}{2}$$

in case $\sin\alpha = \sin\beta_1 = \sin\beta_2 = 0$,

$$n_S(t) \geq \left(1 - 2\frac{a}{\pi}\right) n_{\sigma(\alpha,\beta_1,q_1)}(t) - \frac{n+3}{2}$$

in case $\sin\alpha \neq 0$, $\sin\beta_1 = \sin\beta_2 = 0$ *and*

$$n_S(t) \geq \left(1 - 2\frac{a}{\pi}\right) n_{\sigma(\alpha,\beta_1,q_1)}(t) - \frac{n+1}{2}$$

in case $\sin\alpha = 0$, $\sin\beta_1 \neq 0$, $\sin\beta_2 \neq 0$.

In this paper we give a common generalization of most of the results listed above, see Theorem 1.13 below. In many cases it turns out that weaker lower bound of type (1.13) is sufficient. The details are given in the third part of the paper. In the second part we extend the main results of the paper [?] of the first author; we prove uniqueness from knowledge of eigenvalues and norming constants. Our conditions are connected to the closedness of cosine systems. To formulate the results we need the following definition.

For a sequence $\Lambda = \{\lambda_0, \lambda_1, \dots\} \subset \mathbb{R}$, and for a subset $S \subset \Lambda$ we define the cosine system:

(1.15)

$$C(\Lambda, S) = \{\cos(2\sqrt{\lambda_n}x) : n \in \mathbb{N}_0\} \cup \{x\cos(2\sqrt{\lambda_n}x) : \lambda_n \in S\}.$$

If we are given three types of data: a set of eigenvalues, norming constants and the potential on the part of the interval, we can formulate the following theorem:

Theorem 1.11. *Let $1 \le p \le \infty$, $0 \le a < \pi$, $q_1, q_2 \in L_1(0,\pi)$, $q_1 = q_2$ a.e. on $(0,a)$, $q_1 - q_2 \in L_p(a,\pi)$, $\sin\beta_1 \ne 0$, $\sin\beta_2 \ne 0$ and*

$$\Lambda = \{\lambda_n, \lambda_n \in \sigma(\alpha_n, \beta_1, q_1) \cap \sigma(\alpha_n, \beta_2, q_2), n \in \mathbb{N}_0\}$$

Suppose that $\lambda_n \not\to -\infty$ are different real numbers and $\tau(\lambda_n, \alpha_n, q_1) = \tau(\lambda_n, \alpha_n, q_2)$ if $\lambda_n \in S$ for a subset $S \subset \Lambda$. If $C(\Lambda, S)$ is closed in $L_p(0, \pi - a)$ then $\tan\beta_1 = \tan\beta_2$ and $q_1 = q_2$ on $(0,\pi)$ a.e.

In case of Dirichlet boundary condition the closedness property of the modified cosine system gives an optimal condition.

Theorem 1.12. *Let $1 \le p \le \infty$, $0 \le a < \pi$, $q_1, q_2 \in L_1(0,\pi)$, $q_1 = q_2$ a.e. on $(0,a)$, $q_1 - q_2 \in L_p(0,\pi)$, $\sin\beta_1 = \sin\beta_2 = 0$, let Λ and S be defined as above, $\lambda_n \not\to -\infty$, $\lambda_n \ne \lambda_m$. Let $\mu \ne \pm\sqrt{\lambda_n}$, $\mu \in \mathbb{R}$. Then the system $C(\Lambda, S) \cup \{\cos(2\sqrt{\mu}x)\}$ is closed in $L_p(0, \pi - a)$ if and only if $q_1 = q_2$ on $(0,\pi)$ a.e.*

We can replace the knowledge of finitely many eigenvalues (and norming constant) by the knowledge of the derivatives of q in a. Let us define the common counting function of the eigenvalues and norming constant by:

$$m(t) = 2n_\Lambda(t^2) + 2n_S(t^2). \tag{1.16}$$

In this case we can give the following sufficient condition.

Theorem 1.13. *Let $q_1 = q_2$ a.e. on $(0,a)$ and suppose that for some $\delta_0 > 0$, $k \ge 0$ and $1 \le p < \infty$, $1/p + 1/p' = 1$ we have $q_1 - q_2 \in W^{k,p}([a, a+\delta_0))$ and $(q_1 - q_2)^{(i)}(a) = 0$, $i = 0, 1, \dots k-1$. Consider some common eigenvalues $\lambda_n \in \sigma(\alpha_n, \beta_1, q_1) \cap \sigma(\alpha_n, \beta_2, q_2)$, $n \ge 0$ $\lambda_n \not\to -\infty$, $\lambda_n \ne \lambda_m$ and $\tau(\lambda_n, \alpha_n, q_1) = \tau(\lambda_n, \alpha_n, q_2)$, $\lambda_n \in S$ for* *some* $S \subset \Lambda = \{\lambda_0, \lambda_1, \dots\}$. *If there exists a sequence* $R_i \to \infty$ *such that*

(1.17)
$$\limsup_{i\to\infty}\left[\int_0^{R_i}\frac{m(t)}{t}dt - 4\left(1-\frac{a}{\pi}\right)R_i + \left(k+\frac{1}{p'}\right)\ln R_i\right] > -\infty$$

in case $\sin\beta_1 \neq 0$, $\sin\beta_2 \neq 0$ *and*

(1.18)
$$\limsup_{i\to\infty}\left[\int_0^{R_i}\frac{m(t)}{t}dt - 4\left(1-\frac{a}{\pi}\right)R_i + \left(k+2+\frac{1}{p'}\right)\ln R_i\right] > -\infty$$

in case $\sin\beta_1 = \sin\beta_2 = 0$

then $q_1 = q_2$ *a.e. on* $(0,\pi)$ *and* $\tan\beta_1 = \tan\beta_2$.

If $p = \infty$ *then we suppose additionally that* $q_1, q_2 \in C^k([a, a+\delta_0))$, *and* $(q_1 - q_2)^{(i)}(a) = 0$ *if* $i = 0, \dots k$. *In this case* $\frac{1}{p'} = 1$ *and the same conclusions hold.*

2. Proofs

Proposition 2.1. [?] *Let us denote* $v'(\lambda, x) = \frac{d}{dx}v(\lambda, x)$ *and let*

$$\omega(\lambda) = \sin\alpha v'(\lambda, 0) + \cos\alpha v(\lambda, 0), \tag{2.1}$$

for an arbitrary $\lambda \in \mathbb{C}$. *Then* $\lambda \in \sigma(\alpha, \beta, q)$ *if and only if* λ *is a real zero of* ω. *If* $\lambda \in \sigma(\alpha, \beta, q)$ *then*

$$\frac{\partial\omega}{\partial\lambda} = \kappa(\lambda, \beta, q)\tau(\lambda, \alpha, q). \tag{2.2}$$

An analogue of the following Lemma is proved in the paper of G. Wei and H. Xu [?], Lemma 4.3 for a Sturm-Liouville operator with different type of boundary conditions.

Proposition 2.2. *Denote* $v_i(x, \lambda)$, $i = 1, 2$ *the functions defined by* β_i *and* q_i *in (1.8), (1.9) and (1.10). Suppose* $q_1 = q_2$ *a.e. on* $(0, a)$. *Let*

$$F(x, \lambda) = v_2(\lambda, x)v_1'(\lambda, x) - v_2'(\lambda, x)v_1(\lambda, x), \quad F(\lambda) = F(a, \lambda).$$

Then $F(\lambda) = 0$ *for a real* λ *if and only if there exists an* α *with* $\lambda \in \sigma(\alpha, \beta_1, q_1) \cap \sigma(\alpha, \beta_2, q_2)$. *If* $\lambda \in \sigma(\alpha, \beta_1, q_1) \cap \sigma(\alpha, \beta_2, q_2)$ *then* $\tau(\lambda, \alpha, q_1) = \tau(\lambda, \alpha, q_2)$ *if and only if* $\frac{\partial F(\lambda)}{\partial\lambda} = \dot{F}(\lambda) = 0$.

Proof. If λ is in both spectra then the solutions satisfy the same boundary condition in 0, thus $F(\lambda,0)=0$ by definition and then $F(\lambda)=0$ follows from the fact that $q_1=q_2$ a.e. on $(0,a)$. For general $\lambda\in\mathbb{C}$ we have (omitting arguments if obvious)

$$\frac{\partial F}{\partial x}=v_2v_1''-v_2''v_1=(q_1-q_2)v_1v_2 \tag{2.3}$$

which implies $F(0,\lambda)\equiv F(\lambda)$. If $\sin\alpha\neq 0$, then by (2.1) $v_2(0)\omega_1-v_1(0)\omega_2=\sin\alpha F(0)=\sin\alpha F$. If $\lambda\in\sigma(\alpha,\beta_1,q_1)\cap\sigma(\alpha,\beta_2,q_2)$ then $\omega_1=\omega_2=0$ thus

$$\begin{aligned}\sin\alpha\dot F(\lambda)&=v_2(0)\dot\omega_1-v_1(0)\dot\omega_2=v_2(0)\kappa_1\tau_1-v_1(0)\kappa_2\tau_2\\&=\kappa_2\kappa_1(\tau_2u_2(0)-\tau_1u_1(0))=\kappa_2\kappa_1\sin\alpha(\tau_1-\tau_2)\end{aligned}$$

which is zero if and only if $\tau_1=\tau_2$.

In case of Dirichlet boundary condition, that is $\sin\alpha=0$ we may suppose $\omega_i=v_i(0)$ hence $F=F(0)=\omega_2v_1'(0)-\omega_1v_2'(0)$ and then

$$\dot F=\dot\omega_2v_1'(0)-\dot\omega_1v_2'(0)=\kappa_2\tau_2\kappa_1u_1'(0)-\kappa_1\tau_1\kappa_2u_2'(0)=\kappa_2\kappa_1\cos\alpha(\tau_1-\tau_2)$$

which is zero if and only if $\tau_1=\tau_2$.

□

The proof of Theorem 1.11

Proof. As we have seen, the common eigenvalues λ_n are zeros of $F(\lambda)$ which is an entire function of the variable λ. Thus if the λ_n have a finite accumulation point then F is identically zero and then $v_2(0)v_1'(0)\equiv v_1(0)v_2'(0)$. This means that the Weyl-Titchmarsh m-function of q_1, β_1 and q_2, β_2 are identical. By the Marchenko uniqueness theorem [**?**] it follows that $q_1=q_2$ a.e. and $\tan\beta_1=\tan\beta_2$. So in the remaining part of the proof we suppose that λ_n have no finite accumulation point, and consequently there is a subsequence tending to $+\infty$. Integrating (2.3) in x gives by (1.9) and (1.10)

$$F(\lambda)=\int_a^\pi(q_2(x)-q_1(x))v_1(\lambda,x)v_2(\lambda,x)\,dx-\sin(\beta_2-\beta_1). \tag{2.4}$$

Using the Povzner-Levitan integral representation for v_1 and v_2 (see e.g. [**?**]) we can show that there exists a continuous kernel $K(x,t)$

such that

$$F(z^2)+\sin(\beta_2-\beta_1)=\frac{\sin\beta_1\sin\beta_2}{2}\int\limits_a^{\pi}(q_2-q_1)+$$

$$+\int\limits_0^{\pi-a}\cos(2zx)\left[\frac{\sin\beta_1\sin\beta_2}{2}(q_2-q_1)(\pi-x)+\right.$$

$$\left.+\int\limits_x^{\pi-a}K(x,t)(q_2-q_1)(\pi-t)\,dt\right]dx.$$

The verification is a copying of the proof of Lemma 5.3 of [?] with straightforward modifications, so we omit the details. Taking into account common eigenvalues tending to $+\infty$ we get by the Riemann-Lebesgue lemma the formulae

$$\sin(\beta_2-\beta_1)=\frac{\sin\beta_1\sin\beta_2}{2}\int\limits_a^{\pi}(q_2-q_1) \tag{2.5}$$

and

$$F(z^2)=\int\limits_0^{\pi-a}\cos(2zx)A_{q_2}((q_2-q_1)(\pi-x))\,dx$$

where

$$A_{q_2}h(x)=\frac{\sin\beta_1\sin\beta_2}{2}h(x)+\int\limits_x^{\pi-a}K(x,t)h(t)\,dt.$$

Since $F(\lambda_n)=0$ and for $\lambda_n\in S$ $\dot{F}(\lambda_n)=0$, we get from here that $A_{q_2}((q_2-q_1)(\pi-x))\in L_p(0,\pi-a)$ is orthogonal to $C(\Lambda,S)$. Since this system is closed, we get that $A_{q_2}((q_2-q_1)(\pi-x))=0$ a.e. But A_{q_2} is a Volterra operator, thus $q_1=q_2$ a.e and then $\tan\beta_1=\tan\beta_2$, which completes the proof. □

Proof of Theorem 1.12:

Proof. We can suppose again that Λ has no finite accumulation points. Remark that in case of existence of an accumulation point, $C(\Lambda,S)$ is necessarily closed. Indeed, if h is orthogonal to $C(\Lambda,S)$

then the zeros of its cosine Fourier transform H has a finite accumulation point, hence $H \equiv 0$ and then $h = 0$ a.e.
The if part:
As above, we have a continuous kernel $L(x,t,\mu)$ such that

$$- 2(z^2 - \mu^2)F(z^2) = \int_a^\pi (q_2 - q_1) +$$

$$+ \int_0^{\pi-a} \cos(2zx) \left[(q_2 - q_1)(\pi - x) + \int_x^{\pi-a} L(x,t,\mu)(q_2 - q_1)(\pi - t)\, dt \right],$$

see (2.17) in [?]. Since Λ has a subsequence tending to $+\infty$, we get from here

$$\int_a^\pi (q_2 - q_1) = 0 \tag{2.6}$$

and

$$-2(z^2 - \mu^2)F(z^2) = \int_0^{\pi-a} \cos(2zx) A_{q_2}((q_2 - q_1)(\pi - x))\, dx \tag{2.7}$$

where

$$A_{q_2}(h(x)) = h(x) + \int_x^{\pi-a} L(x,t,\mu)h(t)\, dt. \tag{2.8}$$

Now (2.7) yields that $A_{q_2}((q_2 - q_1)(\pi - x))$ is orthogonal to $C(\Lambda, S)$, consequently $A_{q_2}((q_2 - q_1)(\pi - x)) = 0$ a.e. Since A_{q_2} is Volterra, we obtain $q_1 = q_2$ a.e and then $\tan \beta_1 = \tan \beta_2$.
The only if part:
If $C(\Lambda, S)$ is not complete then there exists $0 \neq h \in L_p(0, \pi - a)$ such that

$$H(z) = \int_0^{\pi-a} h(x) \cos 2xz\, dx$$

has zeros at $\pm\mu$, $\pm\sqrt{\lambda_n}$ and $\dot{H}(\lambda) = 0$ for $\lambda \in S$. Using Lemma 2.1 in [?] we see that for every $q_1 \in L_1(0, \pi)$ there exists a function

$q_2 \in L_1(0,\pi)$ for which $q_1 = q_2$ a.e. on $(0,a)$, $q_1 - q_2 \in L_p(a,\pi)$ and for a sufficiently small $\gamma \neq 0$ we have

$$\gamma h(x) = A_{q_2}((q_2 - q_1)(\pi - x)) \text{ a.e. on } (0, \pi - a).$$

Build up the function F from q_1 and q_2; from (2.7) we get

$$\gamma H(z) = -2(z^2 - \mu^2)F(z^2).$$

Now by Proposition 2.2 we see that there exist α_n with $\lambda_n \in \sigma(\alpha_n, \beta_1, q_1) \cap \sigma(\alpha_n, \beta_2, q_2)$ and that $\tau(\lambda_n, \alpha_n, q_1) = \tau(\lambda_n, \alpha_n, q_2)$ for all $\lambda_n \in S$. The proof is complete. □

To prove Theorem 1.13 we need the following asymptotics for $F(z^2)$. Amour, Faupin and Raoux proved this statement in [**?**] assuming $k \leq 2$ and $\sin\beta_1 \neq 0$, $\sin\beta_2 \neq 0$.

Proposition 2.3. *Under the conditions of Theorem 1.13 for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$|F(z^2)| \leq \frac{e^{2|\Im z|(\pi - a)}}{|\Im z|^{k+\frac{1}{p'}}}\left(\varepsilon + ce^{-\delta(\varepsilon)|\Im z|}\right), \quad \Im z \neq 0$$

in case $\sin\beta_1 \neq 0$, $\sin\beta_2 \neq 0$ *and*

$$|F(z^2)| \leq \frac{e^{2|\Im z|(\pi - a)}}{|z|^2|\Im z|^{k+\frac{1}{p'}}}\left(\varepsilon + ce^{-\delta(\varepsilon)|\Im z|}\right), \quad \Im z \neq 0$$

in case $\sin\beta_1 = \sin\beta_2 = 0$. *The constant* c *does not depend on* ε, δ *and* z.

Proof. Consider first the case $\sin\beta_i \neq 0$. Recall the known asymptotic expression

$$v_i(z^2, x) = \sin(\beta_i)\cos(z(\pi - x)) + \mathbf{O}\left(\frac{e^{|\Im z|(\pi - x)}}{|z|}\right) \quad |z| \to \infty$$

uniform in x, see e.g. in [**?**]. Putting this into (2.4) gives

$$F(z^2) = \sin(\beta_1 - \beta_2) + \frac{\sin\beta_1 \sin\beta_2}{2}\int_a^\pi (q_2 - q_1)$$

$$+ \int_a^\pi (q_2(x) - q_1(x))\left[\frac{\sin\beta_1\sin\beta_2}{2}\cos 2z(\pi - x) + \mathbf{O}\left(\frac{e^{2|\Im z|(\pi - x)}}{|z|}\right)\right] dx.$$

We know from (2.5) that the sum of the first two terms on the right is zero. Hence

$$F(z^2) = \int_a^\pi (q_2 - q_1) f_0, \quad f_0(x) = \mathbf{O}\left(e^{2|\Im z|(\pi - x)}\right) \tag{2.9}$$

uniformly in x. Fix a value $0 < \delta < \delta_0$. Since $q_2 - q_1 \in W^{k,p}[a, a+\delta)$ and $q_1^{(i)}(a) = q_2^{(i)}(a)$ for $i = 0, \ldots, k-1$, we make k integrations by parts to obtain

$$\int_a^{a+\delta} (q_2 - q_1) f_0 = \int_a^{a+\delta} (q_2 - q_1)' f_1 = \cdots = \int_a^{a+\delta} (q_2 - q_1)^{(k)} f_k \tag{2.10}$$

where

$$f_{i+1}(x) = \int_x^{a+\delta} f_i.$$

We see by induction on i that

$$f_i(x) = \mathbf{O}\left(\frac{e^{2|\Im z|(\pi - x)}}{|\Im z|^i}\right)$$

uniformly in x. Now

$$\int_a^{a+\delta} |(q_2 - q_1)^{(k)}(x)| e^{2|\Im z|(\pi - x)} dx \le$$

$$\|(q_2 - q_1)^{(k)}\|_{L_p(a,a+\delta)} \|e^{2|\Im z|(\pi - x)}\|_{L_{p'}(a,a+\delta)} \le$$

$$c_0 \|(q_2 - q_1)^{(k)}\|_{L_p(a,a+\delta)} \frac{e^{2|\Im z|(\pi - a)}}{|\Im z|^{1/p'}}.$$

For small δ the L_p-norm of $(q_2 - q_1)^{(k)}$ is small; for $p = \infty$ this follows from the additional information that $(q_2 - q_1)^{(k)}$ is continuous and zero at $x = a$. The above considerations show that for small δ

$$\Big| \int_a^{a+\delta} (q_2 - q_1) f_0 \Big| \le \varepsilon \frac{e^{2|\Im z|(\pi - a)}}{|\Im z|^{k + 1/p'}}. \tag{2.11}$$

On the other hand

$$\Big|\int_{a+\delta}^{\pi}(q_2-q_1)f_0\Big| \le ce^{2|\Im z|(\pi-a-\delta)} \le c_1\frac{e^{2|\Im z|(\pi-a)}}{|\Im z|^{k+1/p'}}e^{-\delta|\Im z|}$$

which proves Proposition 2.3 if $\sin\beta_i \neq 0$. Now if $\sin\beta_1 = \sin\beta_2 = 0$ then we apply the asymptotic formula

$$v_i(z^2,x) = \frac{\sin(z(\pi-x))}{z} + \mathbf{O}\left(\frac{e^{|\Im z|(\pi-x)}}{|z|^2}\right).$$

We substitute it into (2.4) to obtain

$$F(z^2) = \frac{1}{2z^2}\int_a^{\pi}(q_2-q_1)-$$

$$\int_a^{\pi}(q_2-q_1)(x)\left[\frac{\cos 2z(\pi-x)}{2z^2} + \mathbf{O}\left(\frac{e^{2|\Im z|(\pi-x)}}{|z|^3}\right)\right]dx.$$

The first term is zero by (2.6), hence

$$z^2F(z^2) = \int_a^{\pi}(q_2-q_1)f_0, \quad f_0(x) = \mathbf{O}\left(e^{2|\Im z|(\pi-x)}\right)$$

and we proceed in estimating F as above. □

Proof of Theorem 1.13.

Proof. Suppose indirectly that there are potentials $q_1 \neq q_2$ and β_1, β_2 with the properties specified in Theorem 1.13 and that $F(z^2)$ has zeros in $\lambda_n \in \Lambda$ and (at least) double zeros in $\lambda_n \in S$. If $\sin\beta_i \neq 0$ then

$$F(z^2) = \int_0^{\pi-a}\cos(2zx)A_{q_2}((q_2-q_1)(\pi-x))dx.$$

Since $q_1 \neq q_2$ and A_{q_2} is Volterra, F can not be identically zero. Recall the Jensen formula:

Let $f(z)$ be analytic for $|z| < R$, $f(0) \neq 0$. If $n(t)$ is the number of zeros of $f(z)$ in $|z| \leq t$ then for $0 < r < R$

$$\int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\varphi})| d\varphi - \ln |f(0)|.$$

Applying this formula to $F(z^2)$ we get with the notation $m(t) = 2n_\Lambda(t^2) + 2n_S(t^2)$ that

$$\int_0^r \frac{m(t)}{t} dt \leq \frac{1}{2\pi} \int_0^{2\pi} \ln |F(r^2 e^{2i\varphi})| d\varphi + \mathbf{O}(1).$$

Inserting here the estimate of Proposition 2.3 gives

$$\begin{aligned}
\int_0^r \frac{m(t)}{t} dt \leq \frac{1}{2\pi} \int_0^{2\pi} & \Big[2r|\sin\varphi|(\pi - a) - (k + 1/p') \ln(r|\sin\varphi|) \\
& + \ln(\varepsilon + ce^{-\delta r|\sin\varphi|}) \Big] d\varphi + \mathbf{O}(1) \\
& \leq 4r \left(1 - \frac{a}{\pi}\right) - \left(k + \frac{1}{p'}\right) \ln r \\
& + \frac{1}{2\pi} \int_0^{2\pi} \ln \left(\varepsilon + ce^{-\delta r|sin(\varphi)|}\right) d\varphi + \mathbf{O}(1).
\end{aligned}$$

In the last integral $\varepsilon + ce^{-\delta r|sin(\varphi)|}$ is bounded from above, hence

$$\int_{|\sin\varphi|<1/2} \ln \left(\varepsilon + ce^{-\delta r|sin(\varphi)|}\right) d\varphi \leq \mathbf{O}(1).$$

On the other hand

$$\begin{aligned}
\int_{|\sin\varphi|\geq 1/2} \ln \left(\varepsilon + ce^{-\delta r|sin(\varphi)|}\right) d\varphi & \leq \int_{|\sin\varphi|\geq 1/2} \ln \left(\varepsilon + ce^{-\delta r/2}\right) d\varphi \\
& \leq \int_{|\sin\varphi|\geq 1/2} \ln(2\varepsilon) d\varphi \leq 2\pi \ln\varepsilon + \mathbf{O}(1)
\end{aligned}$$

if r is large enough to ensure $ce^{-\delta r/2} < \varepsilon$. Consequently for every $\varepsilon > 0$ we have

$$\int_0^r \frac{m(t)}{t} dt \leq 4r \left(1 - \frac{a}{\pi}\right) - \left(k + \frac{1}{p'}\right) \ln r + \ln\varepsilon + \mathbf{O}(1)$$

for sufficiently large r; in other words,

$$\int_0^r \frac{m(t)}{t} dt - 4r \left(1 - \frac{a}{\pi}\right) + \left(k + \frac{1}{p'}\right) \ln r \to -\infty, \quad r \to \infty$$

in contradiction to the assumptions of Theorem 1.13. If $\sin\beta_1 = \sin\beta_2 = 0$ then by (2.7)

$$-2(z^2-\mu^2)F(z^2) = \int_0^{\pi-a} \cos(2zx)A_{q_2}((q_2-q_1)(\pi-x))\,dx.$$

Suppose indirectly that $q_2 \neq q_1$; then $(z^2-\mu^2)F(z^2)$ is a nontrivial entire functions with zeros $\pm\mu$, $\pm\sqrt{\lambda_n}$ and at least double zeros for $\lambda_n \in S$. We apply the Jensen formula for $(z^2-\mu^2)F(z^2)$, and the above proof can be repeated with $m(t) = 2n_\Lambda(t^2) + 2n_S(t^2) + 2$. The proof of Theorem 1.13 is complete. □

3. Applications

In this section we show that special cases of Theorem 1.13 give sharper results than those in Theorems 1.5, 1.6, 1.10, and in case $p \neq \infty$ Theorems 1.7, 1.8. We also check that Theorems 1.3, 1.4 and 1.9 are special cases of Theorem 1.13. For the verifications we need the following lemmas.

Lemma 3.1. *Let $A > 0$, $B \in \mathbb{R}$, and define the number $t_{A,B} = 1/2$ for irrational A and $t_{A,B} = \frac{s-1}{2s} + \frac{\{Bs\}}{s}$ if $A = \frac{r}{s}$ is rational with $r, s > 0$, $(r,s) = 1$. Here $\{x\} = x - [x]$ is the fractional part of x. Then*

$$\sum_{k=1}^{N} \frac{\{Ak+B\}}{k} = t_{A,B}\ln N + \mathbf{o}(\ln N), \quad N \to \infty. \tag{3.1}$$

If A is rational, the remainder term $\mathbf{o}(\ln N)$ can be substituted by $\mathbf{O}(1)$.

Proof. Let $\nu_k = \{Ak+B\}$ for short and $S_k = \nu_1 + \cdots + \nu_k$. The usual Abelian summation by parts gives

$$\sum_{k=1}^{N} \frac{\nu_k}{k} = \frac{S_N}{N} + \sum_{k=1}^{N-1} \frac{S_k}{k(k+1)} = \sum_{k=1}^{N-1} \frac{S_k}{k(k+1)} + \mathbf{O}(1). \tag{3.2}$$

Now if $A = \frac{r}{s}$ is rational then ν_k is periodic with period s and average

$$\frac{1}{s}\sum_{k=1}^{s}\{Ak+B\} = \frac{1}{s}\sum_{k=1}^{s}\{Ak\} + \frac{\{Bs\}}{s} = \frac{s-1}{2s} + \frac{\{Bs\}}{s} = t_{A,B}.$$

Consequently $S_k = kt_{A,B} + \mathbf{O}(1)$ and this gives from (3.2) that

$$\sum_{k=1}^{N}\frac{\nu_k}{k} = t_{A,B}\ln N + \mathbf{O}(1).$$

If A is irrational then $S_k = k/2 + \mathbf{o}(k)$ see e.g. Pinner [**?**], and then by (3.2)

$$\sum_{k=1}^{N}\frac{\nu_k}{k} = \frac{1}{2}\ln N + \mathbf{o}(\ln N)$$

as asserted. □

Lemma 3.2. *Let $A > 0$, B, $C \in \mathbb{R}$, $\mu_0 \le \mu_1 \le \mu_2 \le \dots$ be real numbers tending to $+\infty$ and let $m(t) = \sum_{\mu_k \le t} 1$.*
a. If $\sqrt{\mu_k} \le Ak + B + \mathbf{O}(1/k)$ holds for all sufficiently large indices k then

$$\int_1^R \frac{2m(t^2)}{t}dt \ge \frac{2}{A}R + \left(1 - 2\frac{B}{A}\right)\ln R + \mathbf{O}(1), \quad R \to \infty.$$

b. If $\sqrt{\mu_k} \le [Ak+B] + C + \mathbf{O}(1/k)$ for large k then

$$\int_1^R \frac{2m(t^2)}{t}dt \ge \frac{2}{A}R + \left(1 - 2\frac{B+C}{A} + \frac{2}{A}t_{A,B}\right)\ln R + \mathbf{o}(\ln R), \quad R \to \infty.$$

The remainder term can be substituted by $\mathbf{O}(1)$ if A is rational.

Proof. a. The increase of μ_k diminish the counting function $m(t)$ so we can suppose $\sqrt{\mu_k} = Ak + B + \mathbf{O}(1/k)$. Shifting the value of one $\sqrt{\mu_k}$ by $\mathbf{O}(1/k)$ results in a changement $\mathbf{O}(1/k^2)$ in the integral, hence we can suppose $\sqrt{\mu_k} = Ak + B$. Define the function $r(t)$ by

$$2m(t^2) = 2\frac{t-B}{A} + r(t).$$

Clearly $2m(\mu_k) = 2m(\mu_{k+1} - 0) = 2(k+1)$, hence $r(\sqrt{\mu_k}) = 2$, $r(\sqrt{\mu_{k+1}} - 0) = 0$ and $r(t)$ is linear in $[\sqrt{\mu_k}, \sqrt{\mu_{k+1}})$. This implies that

$$\int_{\sqrt{\mu_k}}^{\sqrt{\mu_{k+1}}} \frac{r(t)-1}{t}\, dt =$$

$$= \int_{(\sqrt{\mu_k}+\sqrt{\mu_{k+1}})/2}^{\sqrt{\mu_{k+1}}} (r(t)-1)\left(\frac{1}{t} - \frac{1}{\sqrt{\mu_k}+\sqrt{\mu_{k+1}}-t}\right) dt = \mathbf{O}\left(\frac{1}{k^2}\right)$$

Thus

$$\int_1^R \frac{r(t)}{t} dt = \int_1^R \frac{dt}{t} + \mathbf{O}(1) = \ln R + \mathbf{O}(1)$$

and then

$$\int_1^R \frac{2m(t^2)}{t} dt = \int_1^R 2\frac{t-B}{At} dt + \ln R + \mathbf{O}(1) = \frac{2}{A}R + \left(1 - 2\frac{B}{A}\right)\ln R + \mathbf{O}(1).$$

b. As in a. we can suppose that $\sqrt{\mu_k} = [Ak+B] + C$. Let $\sqrt{\mu_k^*} = Ak + B + C$ and let m^* be the corresponding counting function. From a. we know that

$$\int_1^R \frac{2m^*(t^2)}{t} dt = \frac{2}{A}R + \left(1 - 2\frac{B+C}{A}\right)\ln R + \mathbf{O}(1).$$

On the other hand, for $NA \le R < (N+1)A$ we have

$$\int_1^R \frac{2m(t^2) - 2m^*(t^2)}{t}\, dt = \sum_1^N \int_{\sqrt{\mu_k}}^{\sqrt{\mu_k^*}} \frac{2}{t} dt + \mathbf{O}(1) = \sum_1^N 2\ln\frac{\sqrt{\mu_k^*}}{\sqrt{\mu_k}} + \mathbf{O}(1)$$

$$= \sum_1^N 2\frac{\sqrt{\mu_k^*} - \sqrt{\mu_k}}{\sqrt{\mu_k}} + \mathbf{O}(1) = 2\sum_{k=1}^N \frac{\{Ak+B\}}{Ak} + \mathbf{O}(1).$$

Applying Lemma 3.1 the estimate b. follows. □

Lemma 3.3. *Let $\sigma = \sigma(\alpha, \beta; q)$ and consider an infinite subset $S \subset \sigma$.*
a. Suppose that

$$n_S(t) \geq \gamma n_\sigma(t) + \delta \text{ for large } t > 0 \tag{3.3}$$

for some $0 < \gamma \leq 1$ and $\delta \in \mathbb{R}$. Then for non-Dirichlet boundary conditions $\sin\alpha \neq 0$, $\sin\beta \neq 0$ we have

$$\int_1^R \frac{2n_S(t^2)}{t} dt \geq 2\gamma R + (1 + 2\delta + 2\gamma t_{1/\gamma, -\delta/\gamma}) \ln R + \mathbf{o}(\ln R) \quad r \to \infty. \tag{3.4}$$

For one-sided Dirichlet condition

$$\int_1^R \frac{2n_S(t^2)}{t} dt \geq 2\gamma R + (1 + 2\delta - \gamma + 2\gamma t_{1/\gamma, -\delta/\gamma}) \ln R + \mathbf{o}(\ln R) \quad r \to \infty. \tag{3.5}$$

Finally for two-sided Dirichlet conditions $\sin\alpha = \sin\beta = 0$ we have

$$\int_1^R \frac{2n_S(t^2)}{t} dt \geq 2\gamma R + (1 + 2\delta - 2\gamma + 2\gamma t_{1/\gamma, -\delta/\gamma}) \ln R + \mathbf{o}(\ln R) \quad r \to \infty. \tag{3.6}$$

In all cases the remainder $\mathbf{o}(\ln R)$ can be substituted by $\mathbf{O}(1)$ if γ is rational.
b. If instead of (3.3) we only know that

$$n_S(t) \geq \gamma n_\sigma(t) + \delta \text{ for large } t \in S \tag{3.7}$$

then the estimates (3.4), (3.5) and (3.6) of point a. remain valid if we write $\delta + \gamma - 1$ instead of δ.

Remark that an easy way to obtain lower bounds like (3.4), (3.5), (3.6) is to substitute (3.3) into the integral. Our result gives better lower bounds higher by $(1 - \gamma + 2\gamma t_{1/\gamma, -\delta/\gamma}) \ln R$.

Proof. Let $\sigma = \{\lambda_0, \lambda_1, \dots\}$ and $S = \{\lambda_{k_0}, \lambda_{k_1}, \dots\}$. Since $n_S(t)$ is constant in $[\lambda_{k_{i-1}}, \lambda_{k_i})$ while $n_\sigma(t)$ is growing, we see that (3.3) holds for all large t if and only if $n_S(t-0) \geq \gamma n_\sigma(t-0) + \delta$ holds for all large $t \in S$ if and only if $n_S(t) \geq \gamma n_\sigma(t) + \delta - \gamma + 1$ holds for all large $t \in S$. This argument shows that b. follows from a. and that if (3.3) is true then

$i+1 = n_S(\lambda_{k_i}) \geq \gamma n_\sigma(\lambda_{k_i})+\delta-\gamma+1 = \gamma(k_i+1)+\delta-\gamma+1$ for large i

that is, $k_i \leq \frac{i-\delta}{\gamma}$, i.e.

$$k_i \leq \left[\frac{i-\delta}{\gamma}\right]. \tag{3.8}$$

For non-Dirichlet boundary conditions the well-known eigenvalue asymptotics $\sqrt{\lambda_k} = k + \mathbf{O}(1/k)$ gives

$$\sqrt{\lambda_{k_i}} \leq \left[\frac{i-\delta}{\gamma}\right] + \mathbf{O}\left(\frac{1}{i}\right).$$

That is, Lemma 3.2 applied with $A = 1/\gamma$, $B = -\delta/\gamma$ and $C = 0$ yields (3.4). For one-sided Dirichlet condition the asymptotics $\sqrt{\lambda_k} = k + 1/2 + \mathbf{O}(1/k)$ gives

$$\sqrt{\lambda_{k_i}} \leq \left[\frac{i-\delta}{\gamma}\right] + \frac{1}{2} + \mathbf{O}\left(\frac{1}{i}\right)$$

and Lemma 3.2 applies again with $A = 1/\gamma$, $B = -\delta/\gamma$ and $C = 1/2$. Finally for the two-sided Dirichlet conditions we argue the same way using the asymptotics $\sqrt{\lambda_k} = k + 1 + \mathbf{O}(1/k)$. □

Lemma 3.4. *Let $\gamma > 0$, $\delta, \varrho \in \mathbb{R}$ and $t = t_{1/\gamma,-\delta/\gamma}$. Then*
a. If γ is irrational then

$$\delta > \frac{\varrho-\gamma}{2} \Leftrightarrow 2\delta + 2\gamma t > \varrho.$$

b. If $\gamma = s/r$ is rational, $(r,s) = 1$, $r, s > 0$ then

$$\delta > -\frac{\left[r\left(\frac{\gamma-\varrho}{2} + \frac{1}{2r}\right)\right]}{r} \Leftrightarrow 2\delta + 2\gamma t \geq \varrho.$$

In particular

$$\delta \geq \frac{\varrho-\gamma}{2} + \frac{1}{2r} = \frac{\varrho}{2} + \frac{1-s}{2r} \Rightarrow 2\delta + 2\gamma t \geq \varrho.$$

Proof. If γ is irrational then $t = 1/2$ so point a. is obvious. If $\gamma = s/r$ then $t = \frac{s-1}{2s} + \frac{\{-\delta r\}}{s}$, hence

$$2\delta + 2\gamma t = 2\delta + \frac{s-1}{r} + 2\frac{\{-\delta r\}}{r} = 2\delta + \frac{s-1}{r} + 2\frac{-\delta r - [-\delta r]}{r}$$
$$= \frac{s-1-2[-\delta r]}{r}.$$

Hence

$$\frac{s-1-2[-\delta r]}{r} \geq \varrho \Leftrightarrow [-\delta r] \leq \left[\frac{s-1-\varrho r}{2}\right] \Leftrightarrow -\delta r < \left[\frac{s+1-\varrho r}{2}\right]$$
$$\Leftrightarrow \delta > -\frac{\left[\frac{s+1-\varrho r}{2}\right]}{r} = -\frac{\left[r\left(\frac{\gamma-\varrho}{2} + \frac{1}{2r}\right)\right]}{r}.$$

The last statement follows from here applying the inequality $[x] > x - 1$. □

Now we are able to demonstrate that Theorem 1.13 is a common generalization of Theorems 1.3 to 1.10 and in most cases gives a stronger result even in the special situations described there. Introduce the shorthand notation

$$\begin{aligned} (NN) &\text{ if } \sin\alpha \neq 0, \ \sin\beta_i \neq 0,\\ (DN) &\text{ if } \sin\alpha = 0, \ \sin\beta_i \neq 0,\\ (ND) &\text{ if } \sin\alpha \neq 0, \ \sin\beta_i = 0,\\ (DD) &\text{ if } \sin\alpha = 0, \ \sin\beta_i = 0. \end{aligned}$$

Recall that

$$(3.9) \qquad \int_1^R \frac{2n_{\sigma(\alpha,\beta,q)}(t^2)}{t} dt = \begin{cases} 2R + \ln R + \mathbf{O}(1) \text{ in case } (NN) \\ 2R \text{ in case } (ND) \text{ or } (DN) \\ 2R - \ln R \text{ in case } (DD) \end{cases}$$

see the proof of Lemma 3.2 or [**?**].

Checking Theorem 1.3

In the notation of Theorem 1.13 we have $a = \pi/2$, $p = 1$, $k = 0$, $\Lambda = \sigma(\alpha, \beta, q)$ and $S = \emptyset$. In cases (NN) and (DN) we get from (3.9) that

$$\int_0^R \frac{2n_\Lambda(t^2)}{t} dt \geq 2R + \mathbf{O}(1)$$

which implies uniqueness by (1.17) and $\tan\beta$ is also uniquely identified. In cases (ND) and (DD) we have by (3.9)

$$\int_0^R \frac{2n_\Lambda(t^2)}{t}dt \geq 2R - \ln R + \mathbf{O}(1)$$

which implies uniqueness by (1.18). □

Checking Theorem 1.4
This is case (NN), $a = \pi/2$, $p = \infty$, we have $2k$ instead of k and Λ is σ after deleting $k+1$ eigenvalues. From (3.9) we see that

$$\int_0^R \frac{2n_\Lambda(t^2)}{t}dt = 2R + (1 - 2(k+1))\ln R + \mathbf{O}(1)$$

and uniqueness follows from (1.17). □

Improving Theorem 1.5
Instead of

$$n_S(t) \geq 2\left(1 - \frac{a}{\pi}\right) n_\sigma(t) + \frac{a}{\pi} - \frac{1}{2} \text{ for large } t$$

the weaker bound

$$n_S(t) \geq 2\left(1 - \frac{a}{\pi}\right) n_\sigma(t) + \delta \text{ for large } t,$$

$$\delta > \begin{cases} \frac{a}{\pi} - \frac{3}{2} \text{ if } \frac{a}{\pi} \text{ is irrational or } 1 - \frac{a}{\pi} = \frac{s}{2r},\ s+r \text{ is even} \\ \frac{a}{\pi} - \frac{3}{2} - \frac{1}{2r} \text{ if } 1 - \frac{a}{\pi} = \frac{s}{2r},\ s+r \text{ is odd.} \end{cases}$$

is sufficient. Here and in what follows we always suppose that $r > 0$, $s > 0$ and $(r, s) = 1$. Indeed, let $\gamma = 2(1 - a/\pi)$. We have case (NN), $k = 0$, $p = 1$. By (1.17) and (3.4) we need $1 + 2\delta + 2\gamma t > 0$ if γ is irrational and ≥ 0 if γ is rational. Thus $\varrho = -1$ and by Lemma 3.4 we get $\delta > (-1-\gamma)/2 = a/\pi - 3/2$ if γ is irrational and $\delta > -[r((\gamma+1)/2) + 1/2]/r = -[(s + r + 1)/2]/r$ for $\gamma = s/r$. □

Improving Theorem 1.6
The upper estimate in (1.12) can be deleted, the lower estimate can be weakened, namely it is enough to require for large $t \in \mathbb{R}$

(3.10)

$$n_S(t) \geq 2\left(1 - \frac{a}{\pi}\right) n_\sigma(t) + \delta, \delta > \begin{cases} \frac{1}{2p} - 2 + \frac{a}{\pi} \text{ if } \gamma \text{ is irrational} \\ -\frac{[r(-\frac{1}{2p} + 2 - \frac{a}{\pi} + \frac{1}{2r})]}{r} \text{ if } \gamma = \frac{s}{r}. \end{cases}$$

Indeed, we have (NN), $k = 0$, $1 \le p < \infty$, $\gamma = 2(1 - a/\pi)$ hence (1.17) and (3.4) yield $1 + 2\delta + 2\gamma t > 1/p - 1$ for irrational γ and $\ge$ for rational γ. Thus $\varrho = 1/p - 2$ and Lemma 3.4 gives (3.10). Remark that in the original lower bound (1.12) $\delta = 1/(2p) - 1$, see Lemma 3.3 b. That the bound (3.10) is weaker is obvious for irrational γ and can be checked by

$$\frac{1}{2p} - 1 \ge \frac{1}{2p} - 1 + \frac{1-s}{2r} = -\frac{r(-\frac{1}{2p} + 2 - \frac{a}{\pi} + \frac{1}{2r}) - 1}{r} > -\frac{[r(-\frac{1}{2p} + 2 - \frac{a}{\pi} + \frac{1}{2r})]}{r}$$

for rational γ. □

Improving Theorems 1.7 and 1.8 in case $p \ne \infty$

The statement can be extended from $k = 0, 1, 2$ to every $k \in \mathbb{N}_0$, the upper bound in (1.14) can be deleted and the lower bounds in (1.13) and (1.14) can be weakened by

$$n_S(t) \ge 2\left(1 - \frac{a}{\pi}\right) n_\sigma(t) + \delta, \tag{3.11}$$

$$\delta > \begin{cases} \frac{1}{2p} - 2 + \frac{a}{\pi} - \frac{k}{2} & \text{if } \gamma \text{ is irrational} \\ -\frac{[r(-\frac{1}{2p}+2-\frac{a}{\pi}+\frac{k}{2}+\frac{1}{2r})]}{r} & \text{if } \gamma = \frac{s}{r}. \end{cases}$$

That (3.11) implies uniqueness can be checked in the same way as (3.10) in the special case $k = 0$; we apply Lemma 3.4 with $\varrho = -k - 2 + 1/p$. The original lower bounds are $\delta = \frac{1}{2p} - \frac{3}{2} + \frac{a}{\pi} - \frac{k}{2}$ in (1.13) and $\delta = \frac{1}{2p} - 1 - \frac{k}{2} = \varrho/2$ in (1.14), see Lemma 3.3 b. The latter is smaller but the bound in (3.11) is even smaller. This is straightforward for irrational γ and for $\gamma = s/r$ it can be checked by

$$-\frac{[r(\frac{\gamma-\varrho}{2} + \frac{1}{2r})]}{r} < -\frac{\gamma - \varrho}{2} + \frac{1}{2r} = \frac{\varrho}{2} + \frac{1-s}{2r} \le \frac{\varrho}{2}.$$

□

Checking Theorem 1.9

Before the formal proof we show that if $\sigma(\alpha, \beta_1, q_1) = \sigma(\alpha, \beta_2, q_2)$ then for an eigenvalue $\lambda_n \in \sigma(\alpha, \beta_1, q_1)$ we have $\tau(\lambda_n, \alpha, q_1) = \tau(\lambda_n, \alpha, q_2)$ if and only if $\kappa_w(\lambda_n, \beta_1, q_1) = \kappa_w(\lambda_n, \beta_2, q_2)$. Indeed, the function $\omega_1(z)$ defined in (2.1) is an entire function of order 1/2 whose zeros are precisely the eigenvalues $\lambda_n \in \sigma(\alpha, \beta_1, q_1)$. Consequently by the Hadamard theorem

$$\omega_1(z) = c \prod_{n \geq 0} \left(1 - \frac{z}{\lambda_n}\right). \tag{3.12}$$

If $\sin \beta_1 \neq 0$ then by the known asymptotic formulae

$$v_1(x, z) = \sin(\beta_1) \cos(\sqrt{z}(\pi - x)) + O\left(\frac{e^{|\Im \sqrt{z}|(\pi - x)}}{\sqrt{|z|}}\right) \tag{3.13}$$

$$v_1'(x, z) = \sin(\beta_1) \sin(\sqrt{z}(\pi - x))\sqrt{z} + O\left(e^{|\Im \sqrt{z}|(\pi - x)}\right) \tag{3.14}$$

(see e.g.[**?**]) we obtain

$$\omega_1(z) = \sin \alpha \sin \beta_1 \sqrt{z} \sin(\sqrt{z}\pi) + \mathbf{O}(e^{|\Im \sqrt{z}|(\pi)}).$$

This means that in the constant c of (3.12) there is an unknown factor $\sin \beta_1$; in other words $\omega_1 / \sin \beta_1 = \omega_2 / \sin \beta_2$. Now $\dot{\omega}_i = \kappa_i \tau_i$ implies that in cases (NN) and (DN)

$$\frac{\kappa(\lambda_n, \beta_1, q_1)}{\sin \beta_1} \tau(\lambda_n, \alpha, q_1) = \frac{\dot{\omega}_1(\lambda_n)}{\sin \beta_1}$$

$$= \frac{\dot{\omega}_2(\lambda_n)}{\sin \beta_2} = \frac{\kappa(\lambda_n, \beta_2, q_2)}{\sin \beta_2} \tau(\lambda_n, \alpha, q_2).$$

Since $\kappa_w = \frac{\kappa \sin \alpha}{\sin \beta}$ in case (NN) and $\kappa_w = \frac{\kappa}{\sin \beta}$ in case (DN), this shows that $\kappa_{w,1} = \kappa_{w,2}$ if and only if $\tau_1 = \tau_2$. If $\sin \beta_1 = \sin \beta_2 = 0$ then from the asymptotics

$$v_i(x, z) = \frac{\sin(\sqrt{z}(\pi - x))}{\sqrt{z}} + O\left(\frac{e^{|\Im \sqrt{z}|(\pi - x)}}{|z|}\right) \tag{3.15}$$

$$v_i'(x, z) = -\cos(\sqrt{z}(\pi - x)) + O\left(\frac{e^{|\Im \sqrt{z}|(\pi - x)}}{\sqrt{|z|}}\right) \tag{3.16}$$

we obtain $\omega_1 = \omega_2$. Since $\kappa_w = \kappa$ in case (DD) and $\kappa_w = \kappa \sin \alpha$ in case (ND) so again $\kappa_1 = \kappa_2$ if and only if $\tau_1 = \tau_2$. Now return to the proof of Theorem 1.9. This is the case (NN), $a = 0$, $p = \infty$, $2k - 1$ instead of k, $\Lambda = \sigma(\alpha, \beta_1, q_1)$ and S is $\sigma(\alpha, \beta_1, q_1)$ after deleting

$k+1$ elements. Consequently $m(t) = 4n_\sigma(t^2) - 2k - 2$ for large t and then

$$\int_0^R \frac{m(t)}{t} dt = 4R + (2-2k-2)\ln R + \mathbf{O}(1)$$

which implies uniqueness by Theorem 1.13. $\square$

Improving Theorem 1.10

The lower bounds can be weakened by $n_S(t) \geq (1-2a/\pi)n_\sigma(t) + \delta$ for large $t \in \mathbb{R}$ where

$$\delta > \begin{cases} \frac{a}{\pi} - \frac{n+4}{2} \text{ for irrational } \gamma, \\ -\frac{[r(-\frac{a}{\pi} + \frac{n+4}{2} + \frac{1}{2r})]}{r} \text{ for } \gamma = \frac{s}{r} \end{cases}$$

in case (NN),

$$\delta > \begin{cases} -\frac{n+2}{2} \text{ for irrational } \gamma, \\ -\frac{[r(\frac{n+2}{2} + \frac{1}{2r})]}{r} \text{ for } \gamma = \frac{s}{r} \end{cases}$$

in case (DN),

$$\delta > \begin{cases} -\frac{n+4}{2} \text{ for irrational } \gamma, \\ -\frac{[r(\frac{n+4}{2} + \frac{1}{2r})]}{r} \text{ for } \gamma = \frac{s}{r} \end{cases}$$

in case (ND) and

$$\delta > \begin{cases} -\frac{a}{\pi} - \frac{n+2}{2} \text{ for irrational } \gamma, \\ -\frac{[r(\frac{a}{\pi} + \frac{n+2}{2} + \frac{1}{2r})]}{r} \text{ for } \gamma = \frac{s}{r} \end{cases}$$

in case (DD). Indeed, we have $\gamma = 1 - 2a/\pi$. In case (NN) we need that $1 + 1 + 2\delta + 2\gamma t > -n-1$ for irrational γ and $\geq -n-1$ for $\gamma = s/r$. Consequently $\varrho = -n-3$. In case (DN) we need $1 + 2\delta - \gamma + 2\gamma t > (\geq) - n - 1$, hence $\varrho = -n-1-2a/\pi$. In case (ND) we need $1+2\delta-\gamma+2\gamma t > (\geq) -n-3$, i.e. $\varrho = -n-3-2a/\pi$. Finally in case (DD) we need $-1+1+2\delta-2\gamma+2\gamma t > (\geq) - n - 3$ which means that $\varrho = -n-1-4a/\pi$. In all the four cases we get from Lemma 3.4 that

$$\delta > \begin{cases} \frac{\varrho-\gamma}{2} \text{ for irrational } \gamma \\ -\frac{\left[r\left(\frac{\gamma-\varrho}{2} + \frac{1}{2r}\right)\right]}{r} \text{ for } \gamma = \frac{s}{r}. \end{cases}$$

These are weaker bounds than those in Theorem 1.10. This is obvious for irrational γ and for $\gamma = s/r$ it follows from $-[r((\gamma -$

$\varrho)/2 + 1/(2r))]/r < (\varrho - \gamma)/2 + 1/(2r) = \varrho/2 + (1 - s)/(2r)$. The proof is complete. □

References