(1)

$$\underline{\underline{A}} = \begin{bmatrix} 6 & 1 \\ 8 & 0 \\ 0 & 3/5 \end{bmatrix} \qquad a_{11} \text{ \& } a_{21}\text{-re Givens:} \qquad c = \frac{6}{\sqrt{6^2+8^2}} = \frac{3}{5} \qquad s = \frac{-8}{\sqrt{6^2+8^2}} = \frac{-4}{5}$$

$$\underline{\underline{G}}_1 = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{\underline{G}}_1 \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & - & \frac{4}{5} \\ 0 & & \frac{3}{5} \end{bmatrix} \qquad a_{22} \text{ \& } a_{23}\text{-re Givens}$$

$$c = \frac{+4/5}{\sqrt{\left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}} = \frac{+4/5}{\sqrt{\frac{25}{25}}} = +\frac{4}{5} \qquad s = \frac{3/5}{\sqrt{1}} = \frac{3}{5}$$

$$\underline{\underline{G}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} \qquad \underline{\underline{G}}_2 \underline{\underline{G}}_1 \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 3/5 \\ 0 & & -1 \\ 0 & & 0 \end{bmatrix} = \underline{\underline{R}}$$

$$\underline{\underline{G}}_2 \underline{\underline{G}}_1 \underline{\underline{A}} = \underline{\underline{R}} \rightarrow \underline{\underline{A}} = \overbrace{\underline{\underline{G}}_1^T \underline{\underline{G}}_2^T}^{\underline{\underline{Q}}} \underline{\underline{R}} \quad \text{innen}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & +3/5 \\ 0 & -3/5 & 4/5 \end{bmatrix} = \underline{\underline{G}}_2^T$$

$$\underline{\underline{G}}_1^T = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/5 & -16/25 & -12/25 \\ 4/5 & 12/25 & +9/25 \\ 0 & -3/5 & 4/5 \end{bmatrix} = \underline{\underline{Q}}$$

(2)

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \qquad -1\text{-hez legközelebbi sé: inverz hatványiterációval}$$

$$\left(\underline{\underline{A}} - (-1)\underline{\underline{I}}\right)^{-1} = \begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}^{-1} = -\frac{1}{16}\begin{bmatrix} 0 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/4 & -3/16 \end{bmatrix} \quad \underline{x}_0 = [3; -4] \quad \underline{x}_1 = [-1; \tfrac{3}{2}]$$

$$\underline{x}_2 = \left[\frac{12}{32}; \frac{-17}{32}\right] \qquad \text{Innen } \lambda \approx \frac{\underline{x}_2^T \underline{\underline{A}}\, \underline{x}_2}{\underline{x}_2^T \underline{x}_2} = \frac{[-44, 65][12, -17]}{433} =$$

$$= \frac{-1633}{433} \approx -3.77$$

(3)

$$x_{k+1} = \frac{1}{1+x_k^2} = F(x_k) \qquad \text{mivel } F'(x) = \frac{1}{(1+x^2)2} \cdot (-2x) < 0 \quad I\text{-n és}$$

$$F(0.64) \approx 0.709\ldots \qquad F(0.75) = 0.64 \quad \text{így} \quad F(I) \subset I$$

$$\|F'(x)\|_{\sup}^{[0.64, 0.75]} \leq \frac{-2 \cdot 0.75}{(1+0.64^2)^2} = 0.7549\ldots < 0.76 \quad \text{így } F(x)$$

kontrakció I-n azaz a Banach-féle fixponttétel miatt $\exists!$ fixpontja. Erre $x^* = F(x^*) = \frac{1}{1+x^{*2}} \rightarrow x^*(1+x^{*2}) = 1$ azaz $x^{*3} + x^* - 1 = 0$, azaz tényleg megoldja az egyenletet. Másrészt $f(x) = x^3 + x - 1$-re $f'(x) = 3x^2 + 1 > 0$ így $f(x)$ szig mon nő, azaz tényleg legfeljebb egyetlen megoldása van. Mivel pedig $f(0) = -1$ és $f(1) = 1$ ezért a Bolzano-tétel miatt van megoldása.

Végül $\|\underline{x}_k - \underline{x}^*\| \le \frac{q^k}{1-q}\|x_1 - x_0\| = \frac{0.76^k}{1-0.76}|0.64 - 0.75| < 10^{-4}$

$\Rightarrow k > 30.72\ldots$ azaz legalább 31 lépés kell.

(4) $f(x) = \arcsin(x)$ $[-1,1]$-en 3 Csebisev-alappont

$x_k = \cos\left(\frac{2k+1}{2(n+1)}\pi\right)$ $k = 0,1,2$ $\Rightarrow x_0 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $x_1 = \cos\frac{3\pi}{6} = 0$

$x_2 = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ Innen

$$\begin{array}{ccccc}
\frac{\sqrt{3}}{2} & \frac{\pi}{3} & & & \\
 & & \frac{\pi/3}{\sqrt{3}/2} & & \\
0 & 0 & & 0 & \\
 & & \frac{\pi/3}{\sqrt{3}/2} & & \\
-\frac{\sqrt{3}}{2} & -\frac{\pi}{3} & & &
\end{array}$$

azaz $p(x) = \frac{\pi}{3} + \frac{\pi/3}{\sqrt{3}/2}\left(x - \frac{\sqrt{3}}{2}\right) = \frac{\pi/3}{\sqrt{3}/2}x$

$|\text{Hiba}| \le \frac{f'''(\xi)}{3!} \cdot \frac{1}{2^2}$ $f(x) = \arcsin(x)$ $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

$f''(x) = \frac{-2x}{(\sqrt{1-x^2})^3} \cdot \left(-\frac{1}{2}\right) = \frac{x}{\sqrt{1-x^2}^3}$ $f''' = \frac{\sqrt{1-x^2}^3 - x\frac{3}{2}\sqrt{1-x^2}(-2x)}{(1-x^2)^3}$

$= \sqrt{1-x^2}\,\frac{1-x^2+3x^2}{(1-x^2)^3} = \sqrt{1-x^2}\,\frac{1+2x^2}{(1-x^2)^3}$, így

$|f'''(x)| < 1 \cdot \frac{1+2\left(\frac{1}{2}\right)^2}{1} = 1.5$, azaz $\text{Hiba} \le \frac{1.5}{6\cdot 4} = \frac{1}{16}$

(5)

$$D(h) = \frac{f(x_0+2h) - 2f(x_0) + f(x_0-2h)}{4h^2}$$

$$f(x_0+2h) = f(x_0) + 2h\frac{f'(x_0)}{1!} + 4h^2\frac{f''(x_0)}{2!} + 8h^3\frac{f'''(x_0)}{3!} + 16h^4\frac{f^{(iv)}(\eta)}{4!}$$

$$f(x_0-2h) = f(x_0) - 2h\frac{f'(x_0)}{1!} + 4h^2\frac{f''(x_0)}{2!} - 8h^3\frac{f'''(x_0)}{3!} + 16h^4\frac{f^{(iv)}(\zeta)}{4!}$$

$$\Rightarrow D(h) = \frac{4h^2 f''(x_0) + \frac{2}{3}h^4 f^{(iv)}(\xi)}{4h^2} = f''(x_0) + - h^2 f^{(iv)}(\xi)$$

azaz $D(h)$ a második derivált másodrendű közelítését adja ha $f \in C^4$.

(6) $\int_0^1 \ln(3x+1)\,dx$ Összetett trapéz hibája $\le \frac{M_2}{12}h^2(b-a)$ itt

M_2-hez $f(x) = \ln(3x+1)$ $f'(x) = \frac{3}{3x+1}$ $f''(x) = \frac{9}{(3x+1)^2}$ azaz

$M_2 = 9$, innen $|H| \le \frac{9\cdot h^2\cdot 1}{12} < 10^{-4}$ azaz $h < 0.01155\ldots$ azaz

legalább 87 részre.

Másrészről 3 ekvidisztánssal $\int_0^1 \ln(3x+1)\,dx \approx \frac{1}{3}\left(\frac{1}{2}\ln 1 + \ln 2 + \ln 3 + \frac{1}{2}\ln 4\right)$

$\approx 0.8283\ldots$