A characterization of patch lattices

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To the memory of my friends
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Abstract. In [3] we proved that every planar semimodular lattice is a special gluing, called patchwork of special intervals, called patch lattices, show in Figure 1. In this paper we characterize these patch lattices with invertible $(0,1)$-matrices. This gives hope that the theory if planar semimodular lattices can be traced back to matrix theory.

1. Introduction

Mainly we deal the two-dimensional case, but in many places we discuss the higher dimensional cases too.

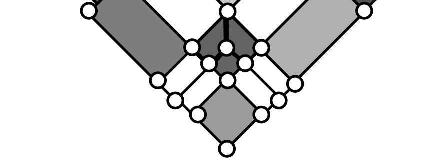

Figure 1. A patchwork in the two-dimensional case

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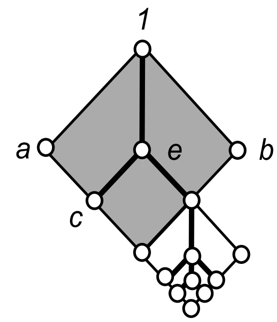

FIGURE 2. Patchwork of two patch lattices

1.1. Source lattices. The *width* $\mathrm{w}(P)$ of a (finite) order P is defined to be $\max\{n$: P has an n-element antichain$\}$. As usual, $\mathbf{J}(L)$ stands for the order of all nonzero join-irreducible elements of L. $\mathbf{Dim}(L) = \mathrm{w}(\mathbf{J}(L))$, consequently 2-dimensional means that the width of the order of join-irreducible elements is two. $\mathcal{C}_n$ denotes the chain $0 < 1 < .... < n-1$ of natural numbers. We define the source in subsection 2.3.

Let us take the lattice N_7, the seven-element semimodular but not modular lattice. This is a source lattice (elementary particle), which is the smallest non distributive building stone of the 2D semimodular lattices.

Remark. *Source lattices* in higher dimensional cases are special join-homomorphic images of the direct powers of $\mathcal{C}_3$ and $\mathcal{C}_2$, whics are the following semimodular lattices:

$$\mathbb{L}_{n,k} = ({\mathcal{C}_3}^{n-k} \times {\mathcal{C}_2}^{k})/\Phi,.$$

Φ is the cover-preserving join-congruence which has only one non-trivial congruence class T (called *beret*), this contains the dual atoms and the unit element. Every non-modular semimodular lattice contains as sublattice a source lattice $\mathbb{L}_{n,k}$. Then $\mathbb{L}_{2,0} \simeq N_7$ and $\mathbb{L}_{3,3} \simeq M_3$.

Two dimensional source lattices are: $\mathbb{L}_{2,0} \cong N_7$, $\mathbb{L}_{2,1} \cong {\mathcal{C}_2}^2$ and $\mathbb{L}_{2,2} \cong \mathcal{C}_2$. In Figure 14 we see $\mathbb{L}_{3,0}$. $\mathbb{L}_{n,k}$ is a filter of $\mathbb{L}_{n,0}$.

1.2. Patch lattices. Patch lattices are the boulding stones (atoms) of the 2D semimodular lattices, see [3].

How can we derive the patch lattices from boolean lattices? This is the *nesting*, which is the following procedure: let L be a semimodular lattice and let I be an interval of L isomorphic to the 2^2-boolean lattice. We call this a 2-cell or *covering square*, see in section 2. On the other hand let us take the lattice N_7 and the four-element sublattice $\{a, b, c, d\}$ (see in Figure 3 and Figure 4,(II) the black marked circles) which is isomorphic to the 2^2-boolean lattice and it is called the *skeleton*, $\mathbf{Sk}(N_7)$ of N_7. There is an isomorphism $\varphi : \mathbf{Sk}(N_7) \longrightarrow I$. We extend this isomorphism to an embedding of N_7 into I. It is easy to extend this order to a semimodular lattice L_1. We can repeat this construction for L_1 and a 2-cell then we get L_2, and so on, we get L_n.

On this way we get from the $L_0 \cong 2^2$ boolean lattice first N_7, these are the *patch lattices*. L_0 is a sublattice of L_n this is the *skeleton* of L_n Let us remark that the dual atoms of the skeleton are dual atoms of the patch lattices, see in Figure 6.

Lemma 1. *Every patch lattice has a skeleton.*

In [3] paper we used for nesting "adding fork to L". Fork is the order $\{c, d, e, 1\}$. In Figure 3 (V) and Figure 4 we see the nesting.

The two-dimensional semimodular lattices can be characterized by $(0,1)$-matrices, $\mathbf{M}_L$, which determines L

The patch lattices are the semimodular lattices which are determined by special non singular (invertible) $(0,1)$-matrices.

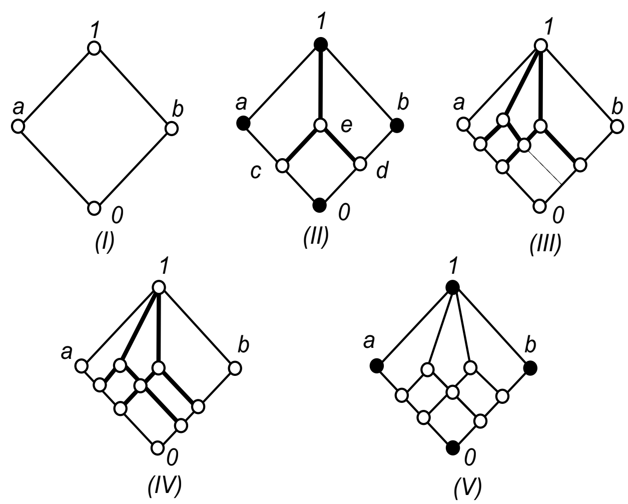

FIGURE 3. The nesting in the 2D case

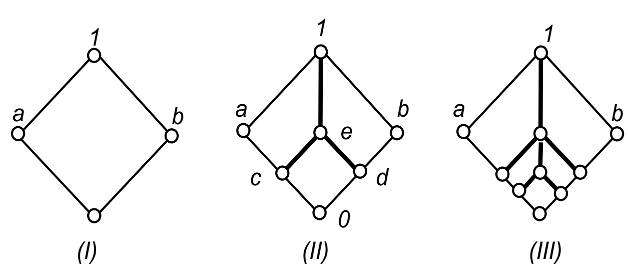

FIGURE 4. The nesting in the 2D case

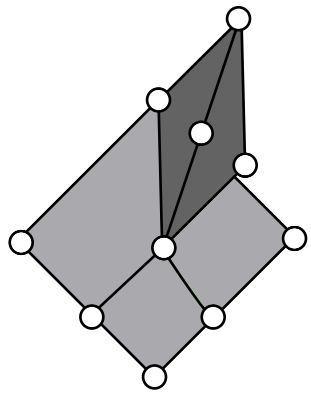

FIGURE 5. A Hall-Dilworth gluing which is not a patchwork

1.2.1. *The building tool: patchwork.* Let L and K be 2-dimensional lattices with the skeletons $\{a \wedge b, a, b, a \vee b\}$ resp. $\{c \wedge d, c, d, c \vee d\}$. The Hall-Dilworth gluing of L and K is called *patching* if $L \cap K \subset [a \wedge b, b]$ and $[c, c \vee d]$, (gluing over edges, $[a \wedge b, b]$ and $[c, c \vee d]$ are one-dimensional).

The following **structure theorem** was proved by G. Czédli and E. T. Schmidt [3].

Theorem 1. *Every two-dimensional semimodular lattice is the patchwork of patch lattices.*

The structure of 2D semimodular lattices:

source lattices (elementary particle)
$\Downarrow$ *nesting* (spec. embedding)
patch lattices (atoms)
$\Downarrow$ *patching* (spec. gluing)
semimodular lattices

Remark. Similar theorem holds for *planar* semimodular lattices in this case the lattices M_n are patch lattices of dimension n.

1.3. Rectangular lattices. Rectangular lattices were introduced by Grätzer-Knapp [9] for planar semimodular lattices. This notion is an important tool by the description of planar semimodular lattices. We define the rectangular lattices for arbitrary dimension.

Definition 1. *A* rectangular *lattice L is a finite semimodular lattice in which* $\mathbf{J}(L)$ *is the disjoint sum of chains* C_i.

Geometric lattices are rectangular. In [6] we introduced the *almost geometric lattices* these are lattices in which $\mathbf{J}(L)$ is the disjoint (cardinal) sum of at most two element chains. In the class of finite distributive lattices the rectangular lattices are the the direct products of chains. The lattices $M_3[\mathcal{C}_n]$ are modular, non distributive, rectangular 3D lattices.

To every 2D semimodular lattice L we assign a $(0,1)$-matrix $\mathbf{M}_L$ in which every row/column contains at most one "1" entrie, see subsection 2.4. Let M be a square matrix such that in the last row and last column the entries are zeros. Delete the last row and last column we get the the *restricted matrix* of M^-. Conversely, if N is a square matrix and we add a new last row and last column with zero entries this is called the *augmented matrix* of N^+. If $N = [1]$ then N^+ is the (augmented) matrix of N_7.

Definition 2. *A* patch matrix *is a square* $(0,1)$*-matrix in which every row/column except the last row/column contains exactly one non-zero entrie and in the last row/column all entries are 0.*

Theorem 2. *Let L be a 2D rectangular semimodular lattice. The following three conditions are equivalent:*

(1) is a patch lattice, i.e. a nested four-element boolean lattice,
(2) L has two dual atoms p and q such that $p \wedge q = 0$ *(then* $0, p, q, 1$ *is the skeleton of L),*
(3) $\mathbf{M}_L$ *is a patch matrix*

The restriced matrix of $\mathbf{M}_L$ is a special non-singular (invertible) matrix.
More equivalent conditions see in [3]

Corollary 1. *The number of non-distributive patch lattices of length n is* $(n-2)!$.

In the two and three dimensional cases $\mathbf{dim}(\mathbf{Sk}(\mathbb{L}_{m,0})) = \mathbf{dim}(\mathbb{L}_{m,0})$.

2. Translation from lattice to matrix.

2.1. Cover-preserving join-homomorphism. There is a trivial "representation theorem for finite lattices: each of them is a join-homomorphic image of a finite distributive lattice D. This follows from the fact that the finite free join semilattices with zero are the finite Boolean lattices.

The semimodular lattices are are very special join-homomorphic images of finite distributive lattices.

Theorem 3. *(Manfred Stern's theorem,* [13]*) Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of finite chains,* $D = C_1 \times C_2 \times ... \times C_k$.

In recent years it was found that this theorem has many interesting consequences. The direct product of n-chains can be considered as an **n-dimensional rectangular shape**, especially a boolean lattice with 2^n-elements is a n -dimensional cube, the direct product of two chains is a plain. This leads to a **geometrical approach** of the semimodular lattices, [12].

In a semimodular lattice the maximal chains have the same length. Assume that L is semimodular lattice and $a, b, u, v \in D, L = \varphi(D)$ and $u \leq a \prec b \leq v$.

In this case φ has a special property. If E is a maximal chain of between u and v and $a, b \in E$, $\varphi(a) = \varphi(b)$ and F is an other maximal chain between u and v , then there exit $c, d \in F, c \prec d$ such that $\varphi(c) = \varphi(d)$. This property is just the cover-preserving property (this is not the usual form).

A planar lattice is called *slim* if every covering square is an interval. Now let L and K be finite lattices. A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving. As usual, $\mathbf{J}(L)$ stands for the order of all nonzero join-irreducible elements of L. For a order P.

In [1] we proved:

Lemma 2. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ).*

Stern's theorem was rediscovered by G. Czédli and E. T. Schmidt [1], see the following theorem (Stern's result was well-hidden in his book):

Theorem 4. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of finite chains, $C_1, C_2, ..., C_n$, these are maximal subchains of L, $n = \mathbf{Dim}(L) = \mathrm{w}(\mathbf{J}(L))$ such that $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup ... \cup C_n$.*

Let us recall the main result from Grätzer and Knapp [9]:

Corollary 2. *(Grätzer and Knapp [9]) Each finite planar semimodular lattice can be obtained from a cover-preserving join-homomorphic image of the direct product of two finite chains and adding doubly irreducible elements to the interiors of covering squares.*

2.2. The grid.

Definition 3. *The* grid *of a semimodular lattice L is $G = C_1 \times C_2 \times ... \times C_n$, where the C_i-s are maximal subchains of L, $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup ... \cup C_n$, $n = \mathbf{dim}(L)$.*

Observe that a grid can be considered as a coordinate system.

By [1] L is the cover-preserving join-homomorphic image of G.

Remark 1. Let $D_1, D_2, ...D_n$, $n = \mathbf{dim}(L)$ be subchains of L such that $\mathbf{J}(L) = D_1 \cup D_2 \cup ... \cup D_n$ then $\underline{G} = D_1 \times ... \times D_n$ is called a (lower) *grid* of L.

2.3. The source. To describe the cover-preserving join- congruences of a distributive lattice G we need the notion of source elements of G. Czédli and E. T. Schmidt [2]. Let Θ be a cover-preserving join-congruence of G.

Definition 4. *An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t$ (Θ) and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v$ (Θ). The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

Lemma 3. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s$ (Θ). If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z$ (Θ).*

Proof. Let s be a source element of Θ then $s \equiv t\ (\Theta)$ for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t\ (\Theta)$. This implies $x \wedge t \not\equiv t\ (\Theta)$. By Lemma 3 we have $x \equiv s\ (\Theta)$.

To prove that $v \not\equiv z\ (\Theta)$, we may assume that $v \prec s$. Take t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely $.x, v, t$. These generate an eight-element boolean lattice in which By the choice of t we know that $v \not\equiv v \wedge t$ (Θ), $x \not\equiv x \wedge t\ (\Theta)$ and $z \not\equiv x \wedge t \wedge v\ (\Theta)$. It follows that $x \not\equiv t\ (\Theta)$, otherwise by the transitivity $x \not\equiv v\ (\Theta)$. □

The following results are proved in [12]. The source $\mathcal{S}$ satisfies an independence property:

Definition 5. *Two elements s_1 and s_2 of a 2D-distributive lattice are s*-independent *if $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$. A subset S is s-independent iff every pair $\{s_1, s_2\}$ is s-independent.*

Remark. In the 2D case every s-independent subset is the source of some cover-preserving join-congruence. In higher dimensional cases this is not thrue, we need an other property too, the shower property [12].

Lemma 4. *Every row/column contains at most one source element.*

The semimodular lattice L is determined by (G, Θ) or $(G, \mathcal{S})$, where $\mathcal{S}$ is an s-independent subset and therefore we write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Determined means, if $L \ncong L'$ then $\mathcal{S} \ncong \mathcal{S}'$ (order isomorphic subsets of G).

Let Θ be a cover-preserving join-congruence of an 2-dimensional grid G and let $\mathcal{S}$ be the source of Θ. Take $\mathcal{S}$ and the set of all lower covers of the source elements ${s'}_i \prec s$ $(i \in \{1, 2, 3\})$. Then we have the following set of primintervals of G:

$$P = \{[{s'}_i, s]|, s \in \mathcal{S}\}.$$

Let $\Theta_{\mathcal{S}}$ be the join congruence generated by this set of primintervals, i.e. for a priminterval $[a, b]$ $a \equiv b\ (\Theta_S)$ if and only if there is a $s \in \mathcal{S}$ priminterval $[{s'}_i, s]$ such that $[a, b]$ is upper perspective to a $[{s'}_i, s]$. Then $\Theta = \Theta_{\mathcal{S}}$ (if $\mathcal{S}$ is an s-independent set then Θ_S.

It is easy to prove that in the 2D case every s-independent subset $\mathcal{S}$ determinate a cover-preserving join-congruence Θ.

Lemma 5. *Let G be a 2-dimensional grid, i.e. the direct product of two chains. Let $\mathcal{S}$ be an s-independent subset of G. Then there exists a cover-preserving join-congruences Θ of G with the source $\mathcal{S}$.*

The meet of two cover-preserving join-congruence is in generally not cover-preserving.

Θ_s denotes the cover-preserving join-congruence determined by s, see in Figure 6. The source of Θ_s is $\{s\}$.

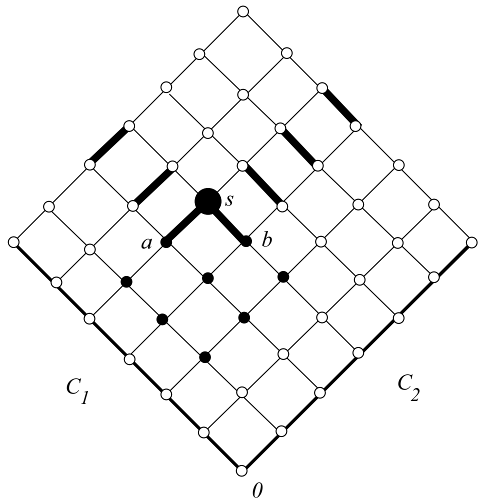

FIGURE 6. The join-congruence Θ_s

2.4. The matrix. Let L be a semimodular lattice. By Theorem 1 we have a grid $G = \mathcal{C}_k{}^n$ and a cover-preserving join-congruence Θ of G such that $G/\Theta \cong L$. In Figure 7 the source $\mathcal{S}$ of Θ has four elements. Put 1 into a cell if its top element is in $\mathcal{S}$, otherwise put zero. What we get is an $n \times n$ matrix, $\mathbf{M}_L$, which determines L (if you like you can turn this grid with 45 degrees to see the matrix in the traditional form). The 7 element semimodular, non modular lattice N_7 has the matrix

$$\mathbf{M}_{N_7} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

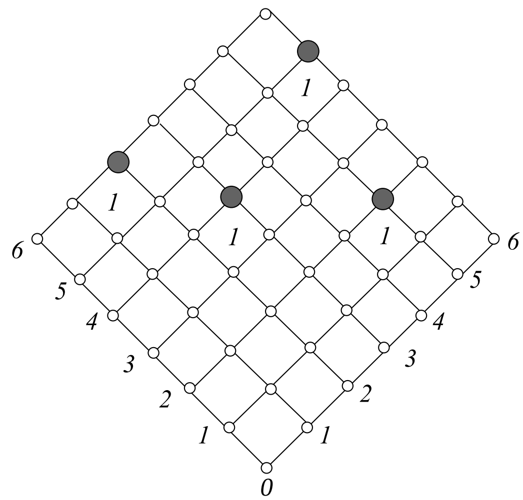

FIGURE 7. A grid and four sours elements

Take the following example. A source and the corresponding matrix is a $n \times n$ $(0,1)$-matrix, where every row/column contains at most 1 entry, the source elements are $s_1 = (6,2), s_2 = (5,6), s_3 = (4,3), s_4 = (2,5)$:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

There is an another possibilitie to get a matrix:

$$\begin{vmatrix} 6 & 5 & 4 & 2 \\ 2 & 6 & 3 & 5 \end{vmatrix}$$

3. The proof of Theorem 2

(1) and (2) are equvivalent, see in [3]

(2) $\Rightarrow$ (3). Let L be a 2D semimodular lattice of lengths n with two dual atoms p and q such that $p \wedge q = 0$, i.e. $0, p, q, 1$ is the skeleton. In Figure 7 you see the $n = 4$ cases. $\mathbf{J}(L)$ is the set of all x $0 < x \leq q, 0 < x \leq p$. Let $C_1 = \{x; 0 < x \leq p\} \cup \{1\}$ and $C_2 = \{x; 0 < x \leq q\} \cup \{1\}$. $(C_1 \approx C_2)$. $G =$ is a grid of L and the cover-preserving join-homomorphism is $\varphi : (x, y) \Rightarrow x \vee y$. Θ denotes the induced cover-preserving join-congruence of G.

$\langle n, 0\rangle \wedge \langle 0, n\rangle = 0$ and $\langle n, 0\rangle \vee \langle 0, n\rangle = 1$. Let $p = \langle n, 0\rangle$ and $q = \langle 0, n\rangle$. The last row rep. last column of the grid doesn't any source element (this would change the order in $\mathbf{J}(G)$). In G/Θ p, q are dual atoms and therefore $\langle n,1\rangle \equiv \langle n,n\rangle$ (Θ) and $\langle 1,n\rangle \equiv \langle n,n\rangle$ (Θ) which means all other rows/columns must contain a source element, with other words the rows/columns contain an entrie 1, i.e. the restricted matrix is non-singular.

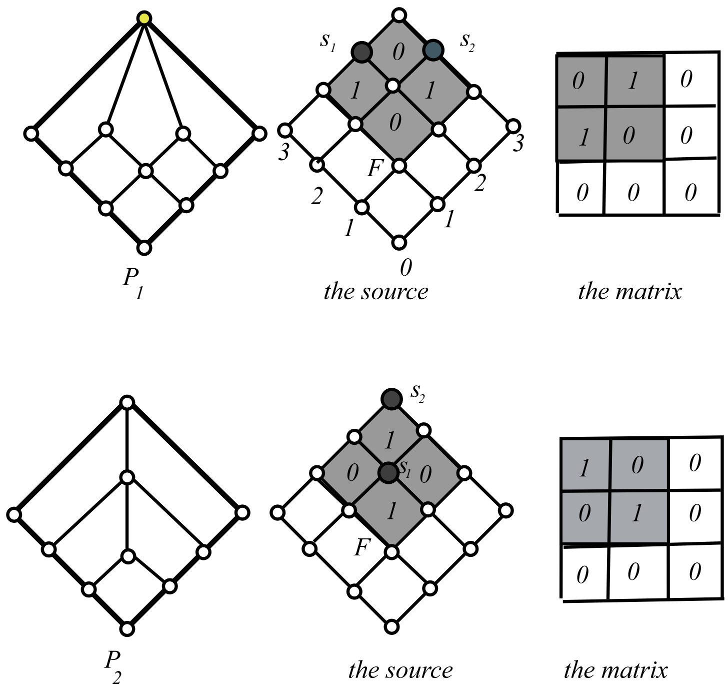

FIGURE 8. Two patch lattices and the matrices

(3) $\Rightarrow$ (2). Let L be a 2D semimodular lattice and assume that $\mathbf{M}_L$ is a patch matrix. The every row rep. column contains "1" entrie, i.e. a source element. If G is the grid then $(n,1) \equiv (n,n)(\Theta)$, $(1,n) \equiv (n,n)(\Theta)$ but $(n,1) \not\equiv (n,0)(\Theta)$, $(n,1) \not\equiv (0,n)(\Theta)$. In the factor lattice $G/\Theta \cong L$ $p=(n,0)$, $q=(0,1)$ are dual atoms and $p \wedge q = 0$.

4. Patch matrices

We consider first $(0,1)$-matrices, in which every row/column has at most one non zero entry, i.e. "1". A $n \times n$ square matrix $M = [a_{i,j}]$ of this kind is *non singular* (or non singular) if every row/column contains exactly one "1". Obviously, every $(0,1)$-non singular matrix is determined by a permutation.

Take a patch matrix, i.e. a $(n+1) \times (n+1)$ matrix N, where the last row and the last column contains only zeros and the remaining $n \times n$ matrix is an non singular matrix M then $N = M_a$ will be called the *augmented* M. If $M = [1]$ then the corresponding augmented matrix is:

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition 6. *A* block *of a matrix $M = [a_{i,j}]$, $1 \le i, j \le n$ is a square submatrix in the form $[a_{i,j}]$, $i \in \{s, s+1, s+2, ..., s+k\}$ and $j \in \{t, t+1, t+2, ..., t+k\}$ for some s, t, k.*

See in Figure 9, the rows/columns of the block are consecutive rows/columns of the given matrix (geometrically it is a convex rectangular).

Definition 7. *A* block matrix *is a system of blocks of a matrix such that the (set theoretical) meet of two blocks does not contain any entry and every entry is in a block.*

Visually, we have a partition of rectangles (blocks). Let us remark that this definition is not the usual definition. In Figure x we have two 4×4-blocks, one 2×2-block and the remaining "0" entries are 1×1-blocks, i.e. *trivial boxes*.

Let M_1 and M_2 two augmented non singular submatrix as blocks of a matrix N. If $M_1 \cap M_2 \neq \emptyset$, i.e. it contains an entry then there are two possibilities, presented in Figure 17 resp. Figure 18 (the blocks can have different sizes). Then $M_1 \cup M_2$ span a block M (convex hull). In all other cases we have a row or column with more then one entry "1". These are the vertical and horizontal sum of M_1 and M_2 (see [12]): $M_1 +_v M_2$ resp. $M_1 +_h M_2$ (these are the generated boxes i.e. the convex rectangular hulls).

We formulate the following easy lemma (the correspondig theorem to Theorem 1):

Lemma 6. *Every $(0,1)$-matrix M, in which every row/column has at most one non zero entry, is a block matrix where the blocks are patch matrices and some 1×1-matrices (with "0" entries), (i.e. it is the patcwork of patch matrices).*

Proof. Let M be a $(0,1)$-matrix in which every row/column has at most one non zero entry. Take the left most 2×2-submatrix M_1 which is an augmented non singular matrices, i.e. has the form given in Figure 24. If this is a maximal augmented non singular matrix then this a block. Otherwise, this is not a maximal augmented non singular matrix then there is an other augmented non singular matrix M_2 such that $M_1 \cap M_2 \neq \emptyset$. This implies that $M_1 +_v M_2$ or $M_1 +_v M_2$ exists. These operations are the *nesting of matrices*. On this way we get a maximal augmented

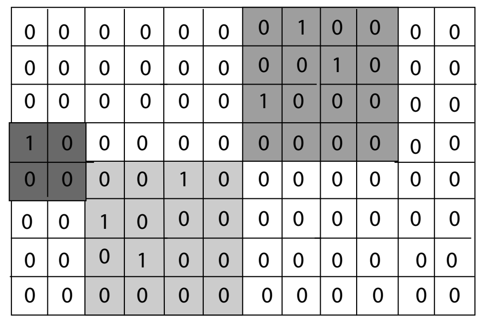

FIGURE 9. A matrix with three non-trivial blocks

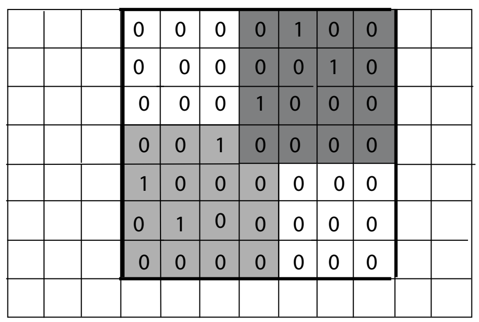

FIGURE 10. Horizontal sum of blocks, $+_h$

non singular matrix. We consider as blocks the maximal augmented non singular $(k \times k)$- matrices. The remaining entries form 1×1 blocks with "0" entries.

Intuitively, we have the "1" entries in the plain, some areas are "density areas" of these entries, these generate a block which is a maximal augmented non singular matrix; the "isolated "1"-s are one-element blocks.

$\square$

Hopefully Lemma 6 allow (planar) semimodular lattices to deal with matrices.

Problem 1. *Establish connection beween Lemma 6 and Theorem 2, prove that Lemma 6 implies Theorem 2.*

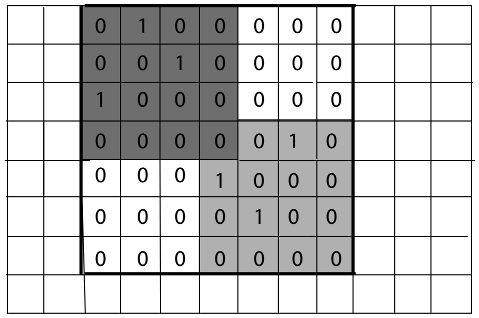

FIGURE 11. Vertical sum of blocks, $+_v$

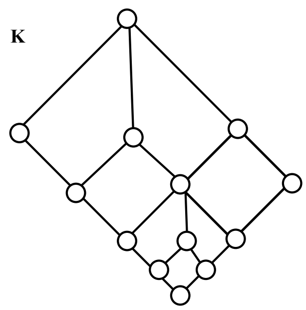

FIGURE 12. The lattice K

In Figre 12 and figure 13 zou can see a block matrix and the corresponding patchwork.

5. Outlook

Theorem 1 is a structure theorem of 2D semimodular lattice. I hope similar theorem holds for all semimodular lattices, see more results in [12].

The skeleton, $\mathbf{Sk}(L)$ of a n-dimensional semimodular lattice is a 2^n-element boolean sublattice, which contains $0, 1$. The "**building stones**" of the structure theorem are special rectangular lattices (in most cases the surface of the diagram is a rectangular shape), we get these from Boolean lattices.

The following 3D rectangular lattices are the patch lattices: $\mathcal{C}^3$, M_3.

Definition 8. *A semimodular lattice is a patchwork lattice if the dual atoms of the skeleton are dual atoms of L.*

Problem 2. *Characterize the patch lattices as nested boolean lattices in the 3D case.*

The direct product $G = C_1 \times C_2 \times C_3$, where C_1, C_2 and C_3 are chains can be considered as a 3D *hypermatrix* (this is a generalization of the matrix to a

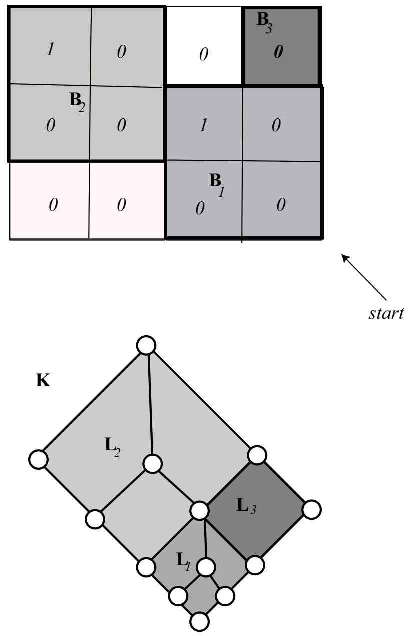

FIGURE 13. Patching of matrices and K

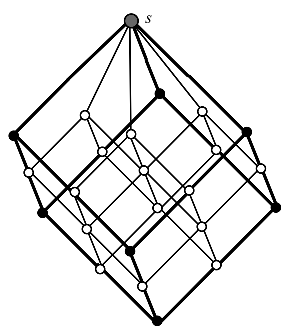

FIGURE 14. 3D source lattice, $\mathbb{L}_{3,0}$.

$n_1 \times n_2 \times n_3$ array of elements: square cuboid), this has a row and two columns. G contains covering cubes, these are called 3-cells. The source elements are top element of the cells, see Figure 8. The 3D hypermatrix of type 2^3 or 3^3 $[a_{i,j,k}]$ is a *source hypermatrix* if $a_{1,1,1} = 1$ and all other entries are zero.

Problem 3. *Characterize the the hypermatrices of patch lattices.*

The "**building tool**" is a kind of gluing, the *patchwork construction*. It is related to the Hall-Dilworth gluing and S-glued sum (Ch. Herrmann [11]), for instance in the 3-dimensional case we glue together cubes (i.e. 2^3 Boolean lattices) over faces, see in Figure 2. Another example is the Rubik cube, the 27 small cubes ("unit cubes") contact with each other along their sides.

Conjecture 1. *Every finite semimodular lattice R is the patchwork of patch lattices.*

A surprising patchwork is the modular lattice $M_3[\mathcal{C}_n]$ where the patch lattices (components) are isomorphic to M_3 or ${\mathcal{C}_2}^2$.

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