A NEW LOOK AT THE SEMIMODULAR LATTICES; A GEOMETRIC APPROACH (RESULTS, IDEAS AND CONJECTURES)

E. TAMÁS SCHMIDT

Contents

1. Introduction 2
1.1. Distributive lattices 2
1.2. Two-dimensional semimodular lattices 3
1.3. The n-dimensional case 5
2. Cover-preserving join-homomorphism 6
3. Dimension concepts 7
4. The source 7
4.1. Shower condition 9
5. The grid and the matrices 10
5.1. The 2D case 10
5.2. The matrix of a cover-preserving join-congruence in the 2D case 12
5.3. The 3D case 13
6. Rectangular lattices 13
7. Source lattices as the elementary particles 14
8. Patch lattices as the atoms 16
8.1. The two-dimensional case. 16
8.2. The three-dimensional case. 17
9. Patchwork 19
9.1. Patching 19
10. Block matrices 21
10.1. The three-dimensional case, hypermatrices. 23
11. The structure theorem (conjecture) 24
12. Appendix: modular lattices. 26

Date: October 13, 2014.

2000 *Mathematics Subject Classification.* Primary: 06C10, Secondary: 06B15.

Key words and phrases. Lattice, semimodular, embedding, gluing, cover-preserving.

References 27

Abstract. The present paper aims to depositing bases of a geometrical approach, we formulate a conjecture, a structure theorem of semimodular lattices, which generalize the the results given in [4] for planar semimodular lattices which asserts, that every planar semimodular lattice is the patchwork of special intervals as show in Figure 1. We examine the structure of the higher-dimensional semimodular lattices and we point out what are the difficulties in higher dimensions. *Our goal is laying the ground.*

The "**building stones**" of the structure theorem are special rectangular lattices (in most cases the surface of the diagram is a rectangular shape), we get these from Boolean lattices, i.e. from n-dimensional cubes.

The "**building tool**" is a kind of gluing, the *patchwork construction.* It is related to the Hall-Dilworth gluing and S-glued sum (Ch. Herrmann [13]), for instance in the 3-dimensional case we glue together cubes (i.e. 2^n Boolean lattices) over faces, see in Figure 2. Another example is the Rubik cube, the 27 small cubes ("unit cubes") contact with each other along their sides.

As technical tool we use special *block matrices.*

1. Introduction

1.1. **Distributive lattices.** A finite distributive lattice D has dimension n if n is the largest natural number such that D contains as sublattice a 2^n-element Boolean lattice. The n-dimensional Boolean lattice is - geometrically - an n-dimensional cube. $\mathcal{C}_n$ denotes the chain $0 < 1 < .... < n-1$ of natural numbers. $(\mathcal{C}_2)^k$ is called the k-dimensional "unit"- cube. The finite distributive lattices have an almost trivial structure theorem. The "building stones" are the "unit" Bolean lattices, i.e. the "unit" cubes and the "building tool" is the Hall-Dilworth gluing. They are glued together by faces.

Theorem 1. *We obtain every n-dimensional finite distributive lattice D if we glue together $2^k, k \leq n$ "unit" cubes by faces.*

See in Figure 2.

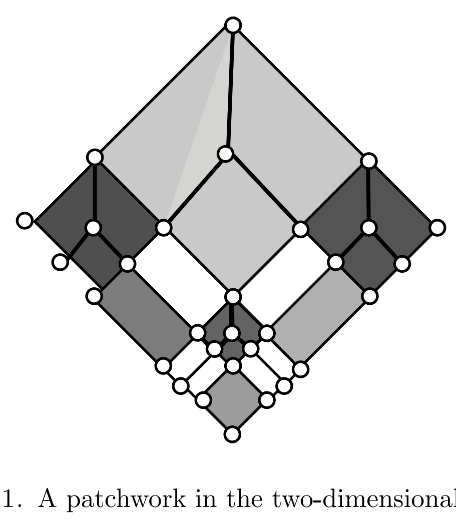

Figure 1. A patchwork in the two-dimensional case

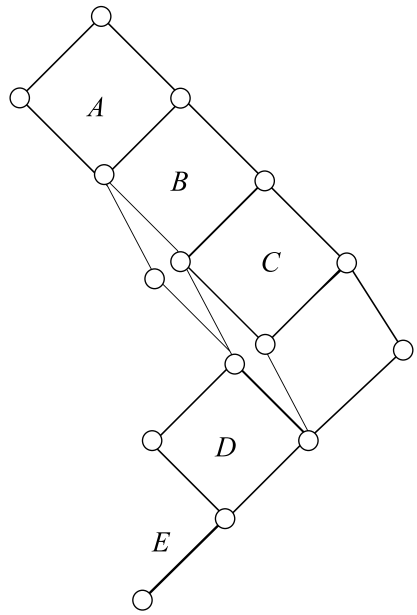

FIGURE 2. A distributive lattice as glued sum of "unit" cubes (3D patchwork)

Is there a similar theorem for semimodular lattices? We guess the answer is yes. Consider first the 2-dimensional semimodular lattices (the dimension is defined in Section 3, two-dimensional is a to planar lattice which does not contain M_3).

1.2. **Two-dimensional semimodular lattices.**

1.2.1. *The boulding stones: patch lattices.* The *width* $\mathrm{w}(P)$ of a (finite) poset P is defined to be $\max\{n\colon P$ has an n-element antichain$\}$. 2-dimensional means that the width of the order of join-irreducible elements is two. How can we derive the patch lattices from boolean lattices? This is the *nesting* (see [4], here we called this procedure adding forks), which is the following procedure: let L be a semimodular lattice and let I be an interval of L isomorphic to the 2^2-boolean lattice. We call this a 2-cell (or covering square, see in section 2). On the other hand let us take the lattice N_7 (the seven-element semimodular but not modular lattice) and the four-element sublattice $\{a, b, c, d\}$ (see in Figure 3 and Figure 4,(II) the black marked circles) which is isomorphic to the 2^2-boolean lattice and it is called the skeleton, $\mathbf{Sk}(N_7)$ of N_7. There is an isomorphism $\varphi : N_7 \longrightarrow I$. We extend this isomorphism to an embedding of N_7 into I. It is easy to extend this poset to a semimodular lattice L_1. We can repeat this construction for L_1 and a 2-cell then we get L_2, and so on. Let us remark that the dual atoms of the skeleton are dual atoms of the patch lattices, see in Figure 6. On this way we get from the 2^2 boolean lattice first N_7. See [4], in this paper we use for nesting "adding fork to L". Fork is the poset $\{c, d, e, 1\}$. In Figure 3 (V) and Figure 4 we see the nesting. Patch lattices are all 2-dimensional semimodular lattices which are obtained by this method.

The two-dimensional semimodular lattices can be characterized by $(0, 1)$-matrices. The patch lattices are the semimodular lattices which are determined by non singular (invertible) $(0, 1)$-matrices.

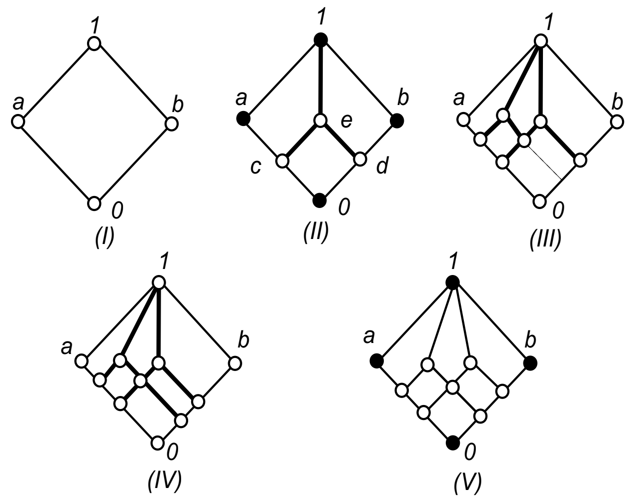

FIGURE 3. The nesting in the 2D case

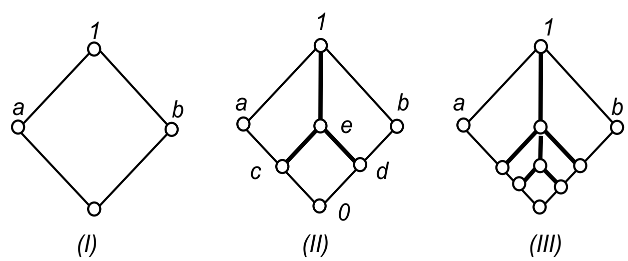

FIGURE 4. The nesting in the 2D case

1.2.2. *The boulding tool: patchwork.* Every 2D patch lattice has a skeleton, if you draw "properly" you see the contour in Figure 7.

Let L and K be 2-dimensional lattices with the skeletons $\{a \wedge b, a, b, a \vee b\}$ resp. $\{c \wedge d, c, d, c \vee d\}$. The Hall-Dilworth gluing of L and K is called *patching* if $L \cap K \subset [a \wedge b, b]$ and $[c, c \vee d]$, (gluing over edges, $[a \wedge b, b]$ and $[c, c \vee d]$ are one-dimensional).

The following **structure theorem** was proved by G. Czédli and E. T. Schmidt [4].

Theorem 2. *Every two-dimensional semimodular lattices is the patchwork of patch lattices.*

Remark. Similar theorem holds for planar semimodular lattices.

Source lattices are special join-homomorphic images of the direct powers of $\mathcal{C}_3$, $\mathbb{L}_{k,0}$, ($\mathbb{L}_{2,0} \simeq N_7$). Then we use a sequence of embeddings of source lattices, the nesting (adding forks in [4]). We get the patch lattices.

We apply patching (a special gluing) of patch lattices. In the two-dimensional case we have the following construction:

source lattices (elementary particle)

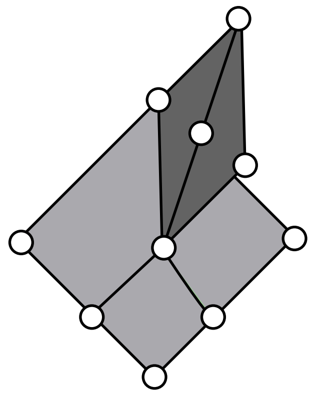

FIGURE 5. A Hall-Dilworth gluing which is not a patchwork

$\Downarrow$ *nesting* (spec. embedding)
patch lattices (atoms)
$\Downarrow$ *patching* (spec. gluing)
semimodular lattices

1.3. **The n-dimensional case.** We formulate the following conjecture: every finite semimodular lattice R is the patchwork of patch lattices (see Conjecture 2 in Section 11).

Here we deal mainly with the three-dimensional case, the higher dimensional cases are similar. We must clarify the concepts: dimension, source lattice, patch lattice, patchwork for arbitrary semimodular lattice.

To every semimodular lattice we assigne a matrix (hypermatrix in the higher dimensional case). **This matrix is the "barcode" of the lattice.** It seems to me that the matrices are the best tools to handle the semimodular lattices.

This paper is the *first step* to prove a structure theorem of semimodular lattices.

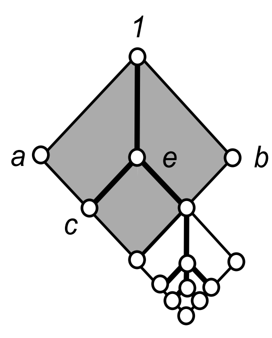

FIGURE 6. Patchwork of two patch lattices

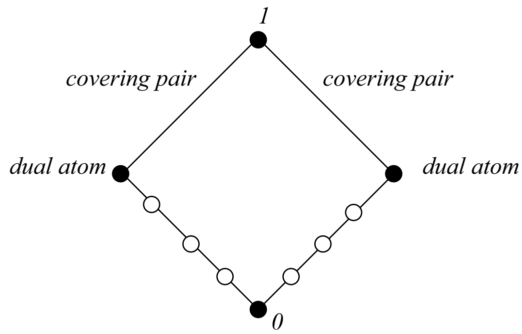

FIGURE 7. The contour of a 2D patch lattice

2. COVER-PRESERVING JOIN-HOMOMORPHISM

There is a trivial "representation theorem" for finite lattices: each of them is a join-homomorphic image of a finite distributive lattice D. We denote this join-homomorphism by φ and Φ is the induced join-congruence of D. This follows from the fact that the finite free join semilattices (with zero) are the finite Boolean lattices.

The semimodular lattices are are very special join-homomorphic images of finite distributive lattices.

Theorem 3. *(Manfred Stern's theorem,* [20]*) Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of finite chains, $D = C_1 \times C_2 \times ... \times C_k$, i.e. $L = \varphi(D)$.*

In recent years it was found that this theorem has many interesting consequences. The direct product of n-chains can be considered as an **n-dimensional rectangular shape**, especially a boolean lattice with 2^n-elements is a n -dimensional cube, the direct product of two chains is a plain. This leads to a **geometrical approach** of the semimodular lattices.

In a semimodular lattice the maximal chains have the same length. Assume that L is semimodular lattice and $a, b, u, v \in D, L = \varphi(D)$ and $u \leq a \prec b \leq v$. In this case φ has a special property. If E is a maximal chain of between u and v and $a, b \in E$, $\varphi(a) = \varphi(b)$ and F is an other maximal chain between u and v , then there exit $c, d \in F, c \prec d$ such that $\varphi(c) = \varphi(d)$. This property is just the cover-preserving property (this is not the usual form).

A sublattice $\{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$ of a lattice is called a *covering sqare* if $a_1 \wedge a_2 \prec a_i \prec a_1 \vee a_2$ for $i = 1, 2$. A planar lattice is called *slim* if every covering square is an interval. Now let L and K be finite lattices. A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving. As usual, $\mathbf{J}(L)$ stands for the poset of all nonzero join-irreducible elements of L. For a poset P, $\mathrm{H}(P)$ denotes the lattice of all hereditary subsets (order ideals) of P.

In [1] we proved:

Lemma 1. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a \ (\Phi)$ and $a \wedge b \not\equiv b \ (\Phi)$ then $a \equiv a \vee b \ (\Phi)$ implies $b \equiv a \vee b \ (\Phi)$.*

Stern's theorem was rediscovered by G. Czédli and E. T. Schmidt [1], see the following two theorems (Stern's result was well-hidden in his book):

Theorem 4. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of* $\mathrm{w}(\mathbf{J}(L))$ *finite chains.*

This direct product is the direct power of a chain C which length is the length of L.

Corollary 1. *The cover-preserving join-homomorphic images of finite distributive lattices are exactly the finite semimodular lattices.*

Theorem 5. *Every finite semimodular lattice L is a cover-preserving join-homomorphic image of the unique distributive lattice D determined by $\mathbf{J}(D) \cong \mathbf{J}(L)$. Moreover, the restriction of an appropriate cover-preserving join-homomorphism from D onto L is a $\mathbf{J}(D) \to \mathbf{J}(L)$ order isomorphism.*

In this theorem D is $H(\mathbf{J}(L))$. Let us recall the main result from Grätzer and Knapp [11]:

Theorem 6. *(Grätzer and Knapp* [11]*) Each finite planar semimodular lattice can be obtained from a cover-preserving join-homomorphic image of the direct product of two finite chains via adding doubly irreducible elements to the interiors of covering squares.*

3. Dimension concepts

The width of $\mathbf{J}(L)$ is called the *dimension* of a semimodular lattice L and will be denoted by $\mathbf{dim}(L)$. If we say L is a n-dimensional semimodular lattice this means $n = \mathbf{dim}(L)$. The ˇ2-dimensional ($\mathbf{dim}(L) = 2$) semimodular lattices are the slim semimodular lattices. (Slim semimodular lattices are the diamond-free planar semimodular lattices.)

An other dimension concept is $\mathbf{Dim}(L)$. $m = \mathbf{Dim}(L)$ is the greatest integer such that L contains a sublattice isomorphic to the 2^m-element boolean lattice. If L is a distributive lattice then $\mathbf{dim}(L) = \mathbf{Dim}(L)$. On the other hand $\mathbf{Dim}(M_3) = 2$ and $\mathbf{dim}(M_3) = 3$. An other example, $\mathbf{Dim}(N_7) = \mathbf{dim}(N_7) = 2$

The third concept is the Kuroš-Ore dimension $\mathbf{dim}_{\mathbf{KO}}(L)$ of L this is the minimal number of join-irreducible elements needed to span the unit element of L.

4. The source

To describe the cover-preserving join- congruences of a distributive lattice G we need the notion of source elements of G. Czédli and E. T. Schmidt [15]. Let Θ be a cover-preserving join-congruence of G.

Definition 1. *An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t \ (\Theta)$ and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v \ (\Theta)$. The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

Lemma 2. *of Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s \ (\Theta)$. If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z \ (\Theta)$.*

Proof. Let s be a source element of Θ then $s \equiv t\ (\Theta)$ for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t\ (\Theta)$. This implies $x \wedge t \not\equiv t\ (\Theta)$. By Lemma 1 we have $x \equiv s\ (\Theta)$.

To prove that $v \not\equiv z\ (\Theta)$, we may assume that $v \prec s$. Take t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely $.x, v, t$. These generate an eight-element boolean lattice in which By the choice of t we know that $v \not\equiv v \wedge t$ (Θ), $x \not\equiv x \wedge t\ (\Theta)$ and $z \not\equiv x \wedge t \wedge v\ (\Theta)$. It follows that $x \not\equiv t\ (\Theta)$, otherwise by the transitivity $x \not\equiv v\ (\Theta)$. □

The following results are proved in [15]. The source $\mathcal{S}$ satisfies an independence property:

Definition 2. *Two elements s_1 and s_2 of a distributive lattice are s-*independent *if $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$. A subset S is s-independent iff every pair $\{s_1, s_2\}$ is s-independent.*

The direct product $G = C_1 \times C_2 \times C_3$, where C_1, C_2 and C_3 are chains can be considered as a 3D *hypermatrix* (this is a generalization of the matrix to a $n_1 \times n_2 \times n_3$ array of elements: square cuboid), this has a row and two columns. G contains covering cubes, these are called 3-cells. the source elements are top element of the cells, see Figure 8.

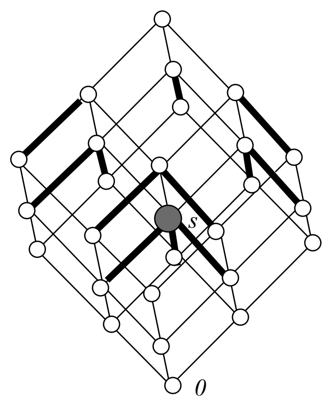

FIGURE 8. The cover-preserving join congruence determined by a source element s

Lemma 3. *Every row/column contains at most one source element.*

The semimodular lattice L is determined by (G, Θ) or $(G, \mathcal{S})$, where $\mathcal{S}$ is an s-independent subset and therefore we write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Determined means, if $L \not\cong L'$ then $\mathcal{S} \not\cong \mathcal{S}'$ (order isomorphic subsets of G).

Let Θ be a cover-preserving join-congruence of an n-dimensional grid G and let $\mathcal{S}$ be the source of Θ. Take $\mathcal{S}$ and the set of all lower covers of the source elements ${s'}_i \prec s$ $(i \in \{1, 2, 3\})$. Then we have the following set of primintervals of G:

$$P = \{[{s'}_i, s]|, s \in \mathcal{S}\}.$$

Let $\Theta_{\mathcal{S}}$ be the join conguence generated by this set of primintervals, i.e. for a priminterval $[a, b]$ $a \equiv b$ (Θ_S) if and only if there is a $s \in \mathcal{S}$ printerval $[s'_i, s]$ such that $[a, b]$ is upper perspective to a $[s'_i, s]$. Then $\Theta = \Theta_{\mathcal{S}}$ (if $\mathcal{S}$ is an s-independent set then Θ_S is generally not a cover-preserving join-congruence).

It is easy to prove that in the 2D case every s-independent subset $\mathcal{S}$ determinate a cover-preserving join-congruence Θ.

Lemma 4. *Let G be a 2-dimensional grid, i.e. the direct product of two chains. Let $\mathcal{S}$ be an s-independent subset of G. Then there exsists a cover-preserving join-congruences Θ of G with the source $\mathcal{S}$.*

The meet of two cover-preserving join-congruence is in generally not cover-preserving.

Θ_s denotes the cover-preserving join-congruence determined by s, see in Figure 8. The source of Θ_s is $\{s\}$.

In the 3-dimensional case the source satisfies the following property:

(3D). If s_1, s_2 are source elements $s_1 \prec s_1 \vee s_2, s_2 \prec s_1 \vee s_2$ and let a be the smallest element such that $s_1 \vee s_2 = (s_1 \wedge s_2) \vee a$ then there is a source element $s_3 \in \mathcal{S}$ such that $s_3 \geq a$.

4.1. **Shower condition.** (Shower) The shower condition: $(1, 1, 1, ..., 1)$ shower head. if $(x_1, 1, 1, ..., 1), x_1 < 1$ and $(1, 2, 1, ..., 1), x_2) < 1$ are source elements, then $(1, 1, 1, .., x_i, .., 1), x_i < 1$ is a source element for all i.

Problem 1. *Characterize the source of a cover-preservig join-congruence of a n-dimensional semimodular lattice.*

In other words: let S be a subset of a grid G (i.e. the direct prodct of chains). Under which conditions is S the source of a semimodular lattice?

Conjecture 1. *A subset $\mathcal{S}$ of G is the source of a cover-preserving join-congruence iff $\mathcal{S}$ is s-independent and satisfies (3D).*

Problem 2. *Prove that $\mathcal{S}$ is the source of a cover-preservig join-congruence Θ iff $\Theta = \bigcup \Theta_s$.*

In Figure 10 we see the non trivial cover-preserving join-congruences of the 3D cube. (D)and (E) are lattice congruences.

By Theorem 1 every finite distributive lattice is the putchwork if cubes. Therefore a 3D grid G is the patchwork of 3D cubes. We want to describe the cover-preseving join-cogruences of G. On Figure 10 we have a row of G. In the given example the first cube has the cover-preserving join-cogruence (A) in Figure 8. This can be uniqualt extend to the given row, we get the CAAA sequence. If (O) denotes the zero congruence then for the following sequences are possible: A..BO, A..AO..O, BO..O, CAAA, CA..AB, DDDDD, EEEEE. On this way we can descibe the cover-preserving join-cogruences.

Figure 10 gives an illustration to the condition (3D) in $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_5$. This is a row of a 3D hypermatrix.

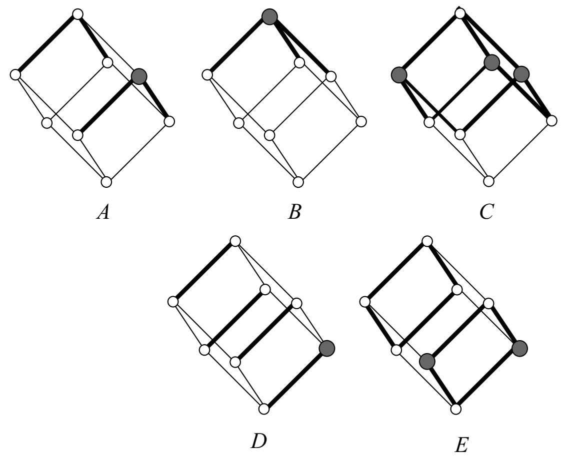

FIGURE 9. The cover-preserving join-congruences of the cube

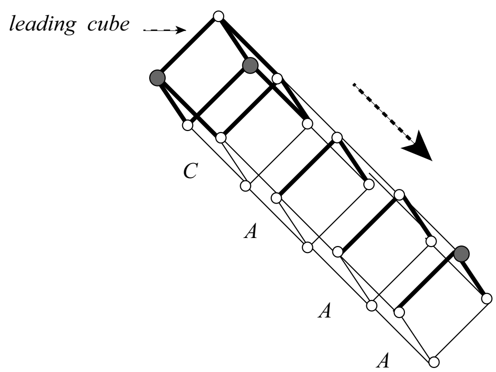

FIGURE 10. A cover-preserving join-congruence of $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_5$

5. The grid and the matrices

5.1. **The 2D case.**

Definition 3. *The (upper)* grid *of a semimodular lattice L is $G = \overline{G} = C_1 \times C_2 \times \ldots \times C_n$, where the C_i-s are subchains of L, $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup \ldots \cup C_n$, $n = \mathbf{dim}(L)$.*

By [1] L is the cover-preserving join-homomorphic image of $\overline{G}$.

Let $D_1, D_2, \ldots D_n$, $n = \mathbf{dim}(L)$ be subchains of L such that $\mathbf{J}(L) = D_1 \cup D_2 \cup \ldots \cup D_n$ then $\underline{G} = D_1 \times \ldots \times D_n$ is called a (lower) *grid* of L.

Observe that a grid can be considered as a coordinate system. We define three important subsets of a grid, The 2D case is given in Figure 11.

Definition 4. *In a grid $G = C_1 \times C_2 \times \ldots \times C_n$ we define the following subsets:*

(1) $\mathbf{I}(G)$ is the order of all inside elements, i.e. coordinates are greather then 1,
(2) $\mathbf{M}(G)$ is the lower margin, the order of all reducible elements where at least one of the coordinates is join-ireducible,

(3) $\mathbf{J_0}(G)$ *denotes the order of all join-irreducible elemens and zero.*

Obviously, $\mathbf{I}(G) \cup \mathbf{M}(G) \cup \mathbf{J_0}(G) = G$. Let $s = (x_1, x_2, ..., x_n)$ be a source element of the grid $G = {\mathcal{C}_k}^n$. If every coordinate $x_i \geq 2$ then $s^* = (x_1 - 2, x_2 - 2, ..., x_n - 2) \in G$. The interval $[s^*, s]$ is isomorphic to ${\mathcal{C}_3}^n$, is called the *forecourt* of s. In Figure 11 the dotted lines represents $\mathbf{M}(G)$, the thick lines represents $\mathbf{J_0}(G)$

This means: if $C_i = c_{i,0} < c_{i,1} < ... < c_{i,k}$ then there is an i such that $x_i = c_{i,1}$, i.e. this coordinate is join-irreducible. If we factorize by Θ_s then the $\mathbf{J}(G)$will be changed. If s is join irreducible then Θ_s is a lattice congruence.

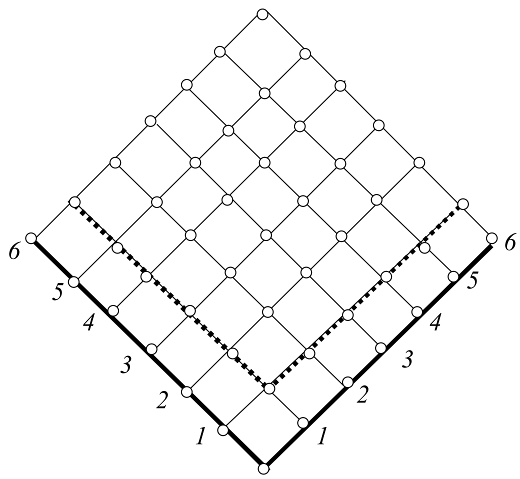

FIGURE 11. The decomposition of a 2D grid

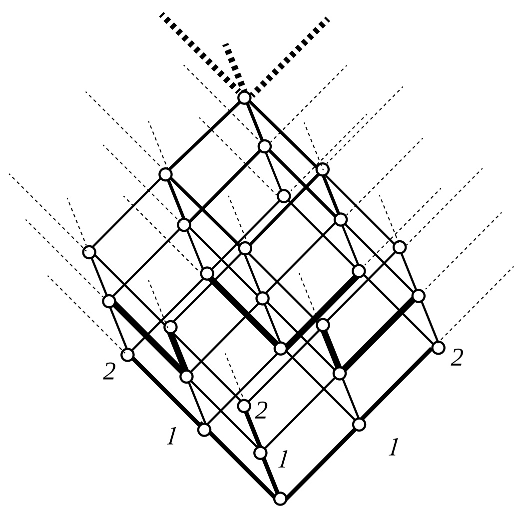

FIGURE 12. The decomposition of a 3D grid

5.2. **The matrix of a cover-preserving join-congruence in the 2D case.** Let L be a semimodular lattice. By Theorem 1 we have a grid $G = {\mathcal{C}_k}^n$ and a cover-preserving join-congruence Θ of G such that $G/\Theta \cong L$. In Figure 12 the source $\mathcal{S}$ of Θ has four elements. Put 1 into a cell if its top element is in $\mathcal{S}$, otherwise put zero. What we get is an $n \times n$ matrix, $\mathbf{M}_L$, which determines L (if you like you can turn this grid with 45 degrees to see the matrix in the traditional form). The 7 element semimodular, non modular lattice N_7 has the matrix

$$\mathbf{M}_{N_7} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

Let M be a square matrix such that in the last row and last column the entries are zeros. Delete the last row and last column we get the the *restricted matrix* of M^-. Conversely, if N is a square matrix and we add a new last row and last column with zero entries this is called the *augmented matrix* of N^+. If $N = [1]$ then N^+ is the (augmented) matrix of N_7.

Take the following example in the 2-dimensional case. A source and the corresponding matrix is a $n \times n$ $(0,1)$-matrix, where every row/column contains at most 1 entry, the source elements are $s_1 = (6,2), s_2 = (3,6), s_3 = (4,3), s_4 = (2,5)$:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

Remark 1. Such a $n \times n$-matrix is determined by two inccreasing sequences $S = (n_1, n_2, ..., n_k)$ and $T = (m_1, m_2, ..., m_k)$ of natural numbers and a permutation $\Pi_{(n-k)}$. This means that in the given matrix the n_1-th, the n_2-th,..., the n_k-th rows

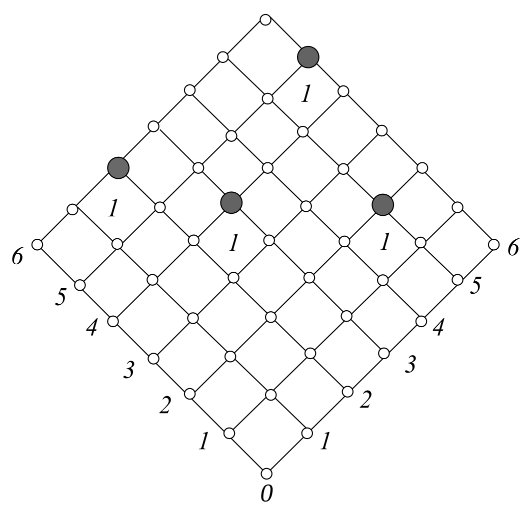

FIGURE 13. A grid and four sours elements

are the zero rows and similary m_1-th, the m_2-th,..., the m_k-th are the zero columns. Delite all zero row and zero column it remains a $(n-k)\times(n-k)$- submatix. In every row/column of this submatrix there is exactly one non-zeto entry (the "1"), i.e. this submatrix is determined by a p n_1-th, the n_2-th,..., the n_k-thermutation. In the given example above the fifth row is the only one zero row, i.e. $S=(4)$, $T=(3)$ and $\Pi_4=(1254)(3)$ in cyclic form.

Particuary important are the lattices $S=(n)$, $T=(n)$ and Π_{n-1}, these are the patch lattices, see in Section 8.

Remark 2. There is an other way to define a matrix using the coordinates of the source elements:

$$\begin{vmatrix} 6 & 5 & 4 & 2 \\ 2 & 6 & 3 & 5 \end{vmatrix}$$

5.3. **The 3D case.** The hypermatrix of M_3 is a $(0,1)$-matrix of type $3\times3\times3$: $[a_{i,j,k}]$, $a_{1,1,1}=1$ and $a_{i,j,k}=0$ otherwise.

A *column* $C(3)_{i,j}$ is $\{a_{i,j,k}; k=1,2,...,n\}$ of a $3\times3\times3$ matrix: $[a_{i,j,k}]$ and similarly,

$C(1)_{i,k}$ is $\{a_{i,j,k}; j=1,2,...,n\}$,

$C(2)_{i,j}$ is $\{a_{j,k}; i=1,2,...,n\}$.

We use $(0,1)$-hypermatrices, where every row/column contains at most one entry 1. See Czédli, Schmidt [3], Czédli [6], Czédli, Ozsvárt, Udvari [7].

$\mathbf{I}(G)=\{(x_1,x_2,x_3)|x_1\geq 2, x_2\geq 2, x_3\geq 2\}$, *inside* of G. G/Θ_s non modular,

$\mathbf{M}(G)=\{(x_1,x_2,x_3)\}$, lower margin of G, one of the coordinates is 0 and one of the coordinates is 1. Θ_s is an ordering relation of $\mathbf{J}(G)$,

$\mathbf{J_0}(G)$ the set of join-irreducile elements and zero. Θ_s is a lattice congruence,

$\mathbf{Mod}(G)=\{x=(x_1,x_2,x_3)|x\notin\mathbf{I}(G), x\notin\mathbf{M}(G), x\notin\mathbf{J_0}(G)\}$ the *modular part* of G, G/Θ_s modular,

See in Figure 12.

6. Rectangular lattices

Rectangular lattices were introduced by Grätzer-Knapp [11] for planar semimodular lattices. This notion is an important tool by the description of planar semimodular lattices. We define the rectangular lattices for arbitrary dimension.

Definition 5. *A* rectangular *lattice L is a finite semimodular lattice in which* $\mathbf{J}(L)$ *is the disjoint sum of chains C_i.*

Geometric lattices are rectangular. In [8] we introduced the *almost geometric lattices* these are lattices in which $\mathbf{J}(L)$ is the disjoint (cardinal) sum of at most two element chains. In the class of finite distributive lattices the rectangular lattices are the the direct products of chains. The lattices $M_3[\mathcal{C}_n]$ are modular, non distributive, rectangular lattices. In Figure 14 is presented $M_3[\mathcal{C}_5]$.

Definition 6. *The* inner skeleton *of a 3D semimodular lattice is sublattice which is an eight-element boolean lattice and contains* 0 *and* 1.

The inner skeleton of L will be denoted by $\mathbf{Sk}(L)$. The inner skeleton of a 2D semimodular lattice is a four-element boolean lattice which contains 0 and 1. That the 3-dimensional lattice R looks like to Figure 16, i.e. to a rectangular shape means that R contains an inner skeleton, in this case this means that $\mathbf{dim}(R)=\mathbf{Dim}(R)$. It is easy to see that $\mathbf{Sk}(R)=\mathbf{Sk}(G_R)=\mathbf{Sk}(\overline{G}_R)$.

Rectangular lattices have a different role, they are packing boxes of semimodular lattices. In [19] we proved, every finite semimodular lattice L can be extended to a rectangular lattice $\widehat{L}$, such that the posets of of join-irreducible elements of L and $\widehat{L}$ have the same width, L and $\widehat{L}$ have the same length.

In the following section we consider special rectangular lattices.

7. Source lattices as the elementary particles

The source lattices are the "smallest" building stones (elementary particles) of the semimodular lattices.

Consider the following semimodular lattices:

$$\mathbb{L}_{n,k} = ({\mathcal{C}_3}^n \times {\mathcal{C}_2}^k)/\Phi, k + n = m$$

which are called the m-dimensional *source lattices.* Φ is the cover-preserving join-congruence of a source lattice which has only one non-trivial congruence class T called *beret*, this contains the dual atoms and the unit element. Every non-modular semimodular lattice contains as sublattice a source lattice $\mathbb{L}_{m,0}$ for some $m > 1$.

Two dimensional: $\mathbb{L}_{2,0} \cong N_7$, $\mathbb{L}_{2,1} \cong {\mathcal{C}_2}^2$ and $\mathbb{L}_{2,2} \cong \mathcal{C}_2$.

Three dimensional: $\mathbb{L}_{3,0} = {\mathcal{C}_3}^3/\Phi$ is non-modular, see in Figure 10, it has a inner skeleton. $\mathbb{L}_{3,1}$, see in Figure 15, it is non-modular and has only an outer skeleton $\mathbb{L}_{3,2}$ is presented in Figure 16 this has a skeleton. The lattice $\mathbb{L}_{3,3}$ is isomorphic to M_3, this means that the source element s is in the margin, $\mathbf{M}(G)$. $\mathbb{L}_{0,3} \cong M_3$.

$\mathbb{L}_{3,3}$ is a dual ideal of $\mathbb{L}_{3,2}$ and $\mathbb{L}_{3,2}$ dual ideal of $\mathbb{L}_{3,1}$ which is a dual ideal of $\mathbb{L}_{3,0}$.

The m-dimensional case, $m > 3$. $\mathbb{L}_{m,0}$ is non modular, $\mathbb{L}_{m,m} \simeq M_m$. In all other cases we have non -modular lattices.

A four dimensional: $\mathbb{L}_{4,4}$ in Figure 19.

Source lattice are the lattices $\mathbb{L}_{m,0}$ and its dual ideals.

Observe that $\mathbb{L}_{m,k}$ is in only one case (!) a modular non-distributive lattice if $m = 3$. $k = m$, which gives M_3.

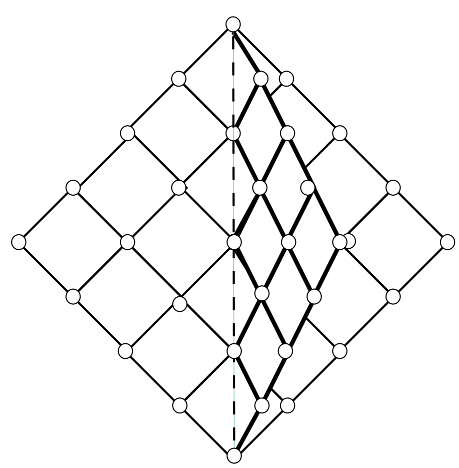

FIGURE 14. $M_3[\mathcal{C}_4]$ a 3D rectangular, modular lattice

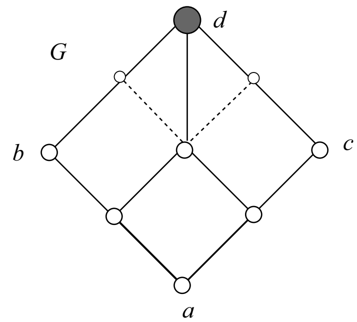

FIGURE 15. N_7

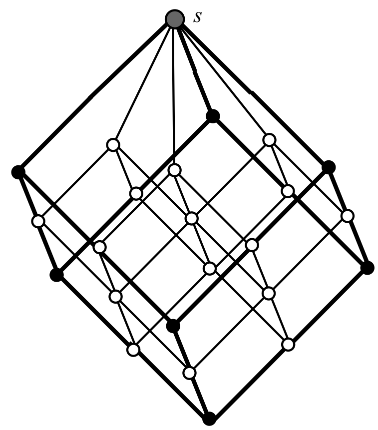

FIGURE 16. The source lattice $\mathbb{L}_{3,0}$ and the inner skeleton

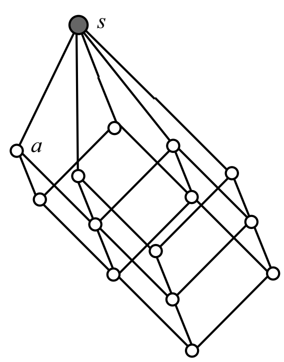

FIGURE 17. $\mathbb{L}_{3,1}$

Let us remark that $\mathbf{dim}(\mathbb{L}_{m,0}) = \mathbf{Dim}(\mathbb{L}_{m,0}) = \mathbf{dim_{KO}}(\mathbb{L}_{m,0})$, but for $M_3 = \mathbb{L}_{3,3}$ this is not true. These lattices have skeletons, $\mathbf{dim}(\mathbf{Sk}(\mathbb{L}_{m,0})) = \mathbf{dim}(\mathbb{L}_{m,0})$.

The restricted hypermatrix of $\mathbb{L}_{n,0}$ is a hypermatrix where $a_{1,1,\ldots,1} = 1$ and all other entries are zero.

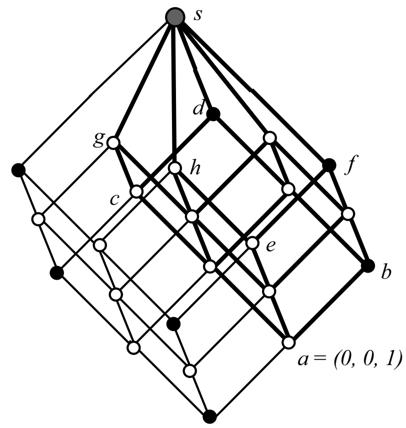

FIGURE 18. $\mathbb{L}_{3,1}$ as a dual ideal of $\mathbb{L}_{3,0}$

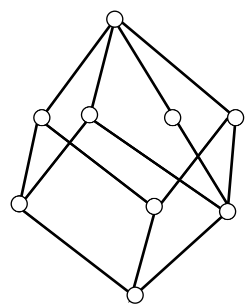

FIGURE 19. $\mathbb{L}_{3,2}$ and the skeleton

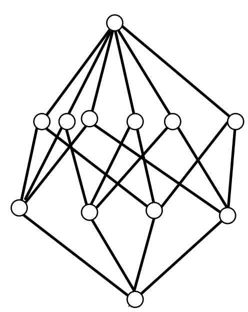

FIGURE 20. $\mathbb{L}_{4,4}$ and the inner skeleton

The source lattice is the image of the forecourt of the source element.

8. Patch lattices as the atoms

8.1. **The two-dimensional case.** The concept of patch lattices was originally introduced in the planar case in [4]. These are nested source lattices, see in subsection 1.2. First of all we consider the matrices of the 2D semimodular lattices.

Take the construction in subsection 1.2 we started with N_7. The matrix of N_7 is the following 2×2 matrix:

$$\mathbf{M}_{N_7} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

This belongs to the lower grid of N_7. The restricted matrix is the 1×1 matrix [1], which is obviousy a non singular (invertible) matrix. The second step is that we take a 2-cell and insert into this interval N_7. Then the new grid is $G = \mathcal{C}_3 \times \mathcal{C}_3$, i.e we have a 4×4 matrix, as in Figure 3 and Figure 4. The corresponding matrices are in Figure 21. These are again non singular matrices of type 3×3.

Lemma 5. *([4]) Let L be a 2D rectangular semimodular lattice. The following three conditions are equvivalent:*

(1) L is a nested boolean lattice,
(2) the dual atoms of the skeleton, a and b are dual atoms of L,
(3) is a square $0, 1$-matrix in which every row/column except the last row/column contains exactly one non-zero entrie and in the last row/column all entries are 0.

We can define the 2D patch lattices as follows:

Definition 7. *A rectangilar lattice L ls a patch lattice if the dual atoms of the skeleton, are dual atoms of L.*

Every source lattice $R = \mathbb{L}_{m,0}$ has a skeleton $\mathbf{Sk}(R)$. The dual atoms of $\mathbf{Sk}(R)$ are dual atoms of R. Modular but not distributive 2D patch lattice does not exist.

8.2. **The three-dimensional case.** There are several ways to introduce the patch lattice in the 3D case, I propose the following:

An 3-dimensional semimodular lattice P is called patch lattice if its hypermatrix every row and column except the last row and last columns contains a non zero entry and satisfies some additional properties (see Problem 1).

We can define the nesting in the 3D case too but it seems very complieted.

This is the nesting, which is the following procedure: let L be a semimodular lattice and let I be an interval of L isomorphic to the 2^n-boolean lattice. We call this an n-cell. On the other hand let us take the source lattice $\mathbb{L}_{n,k}$ and the skeleton, $\mathbf{Sk}(\mathbb{L}_{n,k})$ which is isomorphic to the 2^n-boolean lattice, i.e. there is an isomorphism $\varphi : \mathbf{Sk}(\mathbb{L}_{n,k}) \longrightarrow I$. We extend this isomorphism to an embedding of $\mathbb{L}_{n,k}$ into I. It is easy to extend this poset to a semimodular lattice. We can repeat this construction for L_1 and get L_2, and so on.

M_3 is a 3D patch lattice which is not a nested lattice. The restricted hypermatrix M_3 is the $1 \times 1 \times 1$ hypermatrix with the entry 1.

Problem 3. *Give a description of nesting in the 3D case.*

The description seems to me very complicatid.

Problem 4. *Is M_3 the only one 3D patch lattice which is not a nested boolean lattice?*

Remark. The hypermatrix M_P of a patch lattce P satisfies an additional condition $(3D)$.

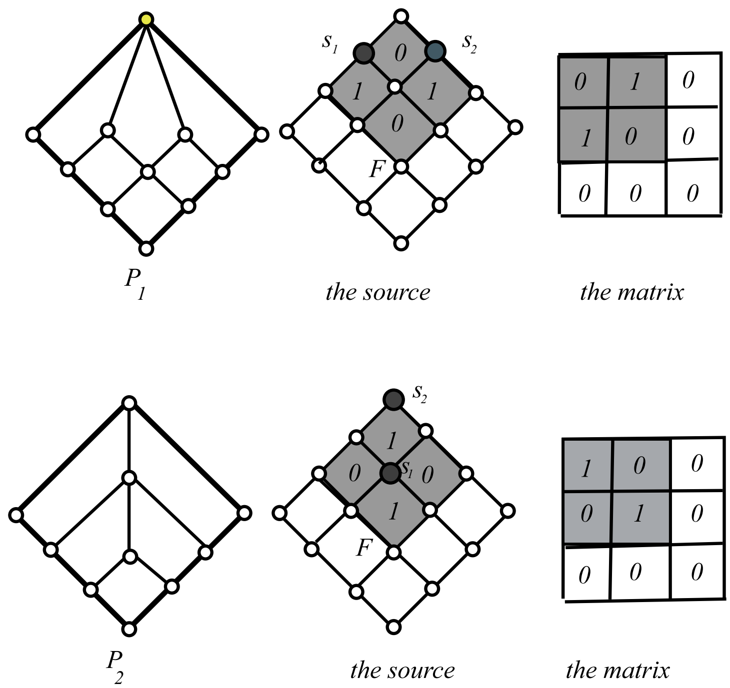

FIGURE 21. Two 2D patch lattices with the sources and the matrices

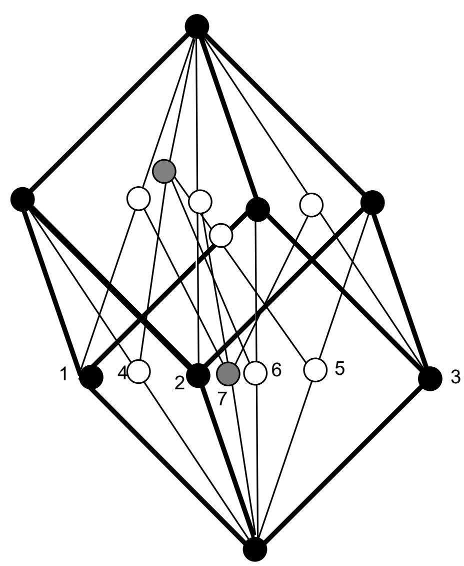

FIGURE 22. The Fano plain

Lemma 6. *The (hyper)matrix M_P of a patch lattice P is a (hyper)patchmatrix.*

Proof. Let P be a patch lattice represented by the source, $P = \mathcal{P}(G, \mathcal{S})$. The dual atoms of the skeleton are dual atoms of P, which means every row/column - except

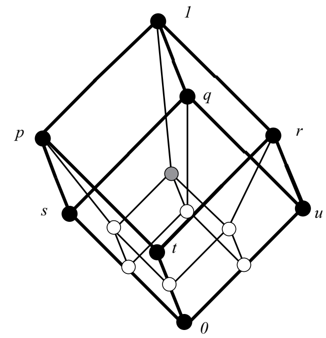

FIGURE 23. A 3D patch lattice the Edelman-Jaison lattice

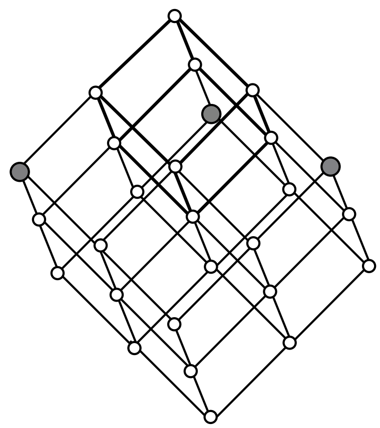

FIGURE 24. The source of the Edelman-Jaison lattice

the last row and last column - contains a source element. Therefore, in the matrix representation every row/column there is an entry 1. □

Remark. Let R be a 3-dimensional rectangular semimodular lattice. If we have 3 disjoint chains C_1, C_2 and C_3 then rectangular means that $\mathbf{J}(R)$ is the disjoint sum of these chains. How does it looks like R? The first answer is, visually we see Figure 25, if you draw "properly". The direct product $G = C_1 \times C_2 \times C_3$ is such a lattice. There is an other lattice of this type: this is $M_3[\mathcal{C}_n]$, see Figure 14 (here as patchwork of covering squares and M_3-s). It is interesting that this is modular.

9. PATCHWORK

9.1. **Patching.** The idea is the following: In Figure 25 are some examples of 3D patchings. The cubes are the skeletons of the patch lattices. The gluing is over subintervalls of the faces. 2D patchwork see in Flgure 1.

To define the patchwork construction (a special gluing) we need a dimension. We have seen in Section 2 there are different dimension concepts in semimodular lattices. We take here $\mathbf{Dim}(L)$.

Let K and L be semimodular lattices, let F be a filter of K, and I be an ideal of L. We consider these as geometric shapes (e.g. cubes), K and I are adhesive

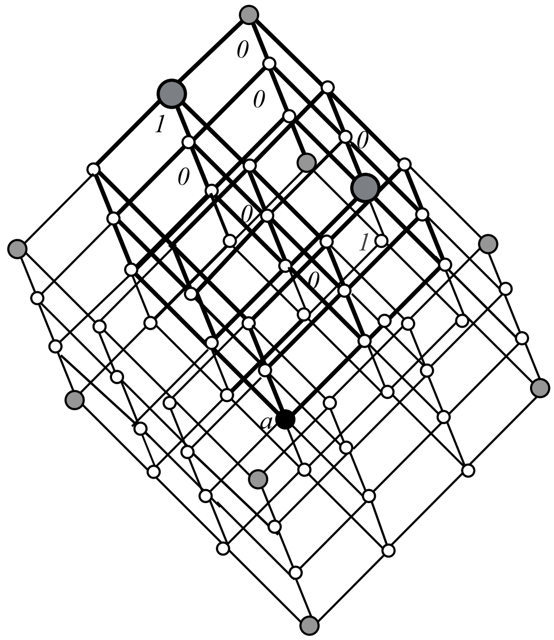

FIGURE 25. A 3D patch lattice with the skeleton and the hypermatrix

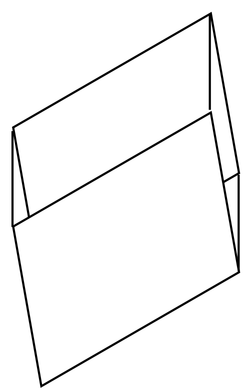

FIGURE 26. The contour of a 3D rectangular semimodular lattice, a cuboid.

faces, which are isomorphic. Then we can form the the lattice G, the well-known Hall-Dilworth gluing of K and L over F and I. Assume that $\mathbf{Dim}(K)$, $\mathbf{Dim}(L)$, $\mathbf{Dim}(F)$, $\mathbf{Dim}(I)$ are defined. We call the gluing G the *patching* of K and L if:

$$\mathbf{Dim}(I) < \min(\mathbf{Dim}(K), \mathbf{Dim}(L)). \tag{Dim}$$

Then G is the *patching* of K and L. See in Figure 2, the cubes are the skeletons (as an example $A \simeq M_3, B \simeq \mathbb{L}_{3,0}$). We consider the general case. Let S be a semimodular lattice. Let $\{B_i\}$ be a system of intervals of S - called blocks if

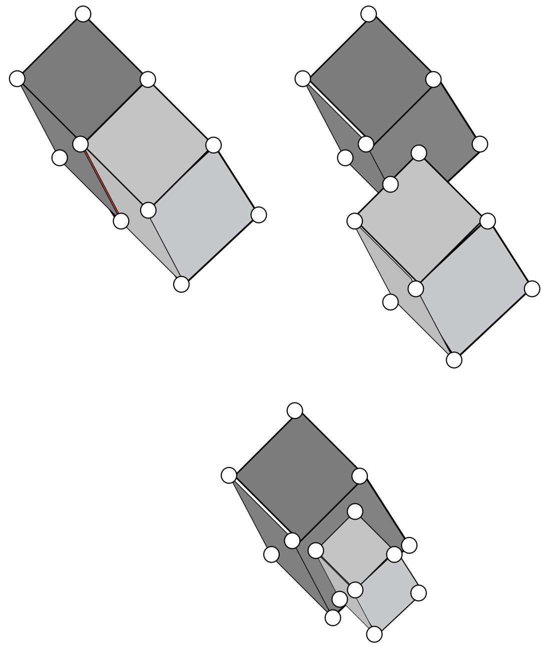

FIGURE 27. Some example of 3D patching

$\bigcup B_i = L$ and if $B_i \cap B_j \neq \emptyset, i \neq j$, then the union, $B_i \cup B_j$ is the Hall-Dilworth gluing satisfying (DID) (i.e. the gluing is via an edge or a face.

If K_1, K_2 and K_3 are face (2-dimensional) of blocks such that (see in Figure 1) $K_1 \cap K_2$ is an ideal of K_2 and similarly $K_1 \cap K_3$ is an ideal of K_3. then this is a *patching system*.

In Figure 6 we see a Hall-Dilworth gluing which is not a patchwork.

10. Block matrices

The 2-dimensional case. We consider first $(0,1)$-matrices, in which every row/column has at most one non zero entry, i.e. "1". A $n \times n$ square matrix $M = [a_{i,j}]$ of this kind is *non singular* (or non singular) if every row/column contains exactly one "1". Obviously, every $(0,1)$-non singular matrix is determined by a permutation.

Take a patch matrix, i.e. a $(n+1) \times (n+1)$ matrix N, where the last row and the last column contains only zeros and the remaining $n \times n$ matrix is an non singular matrix M then $N = M_a$ will be called the *augmented* M. If $M = [1]$ then the corresponding augmented matrix is:

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We don't write the matrices in the usual form, we use the following "table" notation, see in Figure 15.

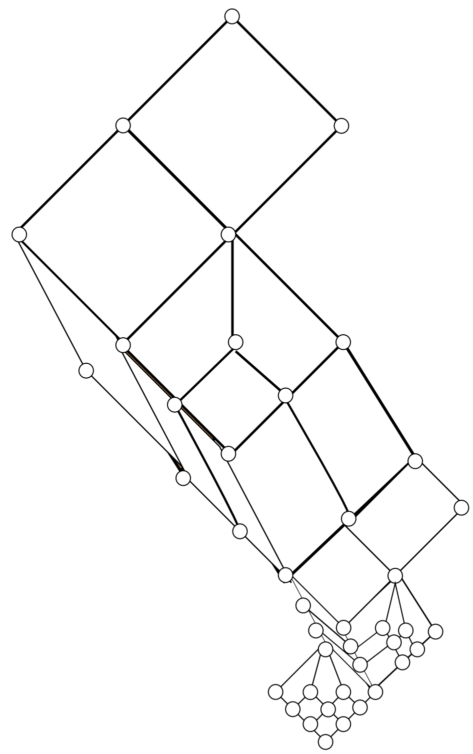

FIGURE 28. A 3-dimensonal patchwork

Definition 8. *A* block *of a matrix* $M = [a_{i,j}]$, $1 \leq i, j \leq n$ *is a square submatrix in the form* $[a_{i,j}]$, $i \in \{s, s+1, s+2, ..., s+k\}$ *and* $j \in \{t, t+1, t+2, ..., t+k\}$ *for some* s, t, k.

See in Figure 25, the rows/columns of the block are consecutive rows/columns of the given matrix (geometrically it is a convex rectangular).

Definition 9. *A* block matrix *is a system of blocks of a matrix such that the (set theoretical) meet of two blocks does not contain any entry and every entry is in a block.*

Visually, we have a partition of rectangles (blocks). Let us remark that this definition is not the usual definition. In Figure 25 we have two 4×4-blocks, one 2×2-block and the remaining "0" entries are 1×1-blocks, i.e. *trivial boxes*.

Let M_1 and M_2 two augmented non singular submatrix as blocks of a matrix N. If $M_1 \cap M_2 \neq \emptyset$, i.e. it contains an entry then there are two possibilities, presented in Figure 17 resp. Figure 18(the blocks can have different sizes). Then $M_1 \cup M_2$ span a block M (convex hull). In all other cases we have a row or column with more then one entry "1". These are the vertical and horizontal sum of M_1 and M_2 (see [16]): $M_1 +_v M_2$ resp. $M_1 +_h M_2$ (these are the generated boxes i.e. the convex rectangular hulls).

We formulate the following easy, almost trivial lemma:

Lemma 7. *Every $(0,1)$-matrix M, in which every row/column has at most one non zero entry, is a block matrix where the blocks are patch matrices and some 1×1-matrices (with "0" entries), (i.e. it is the patcwork of patch matrices).*

0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 29. A matrix with three non-trivial blocks

Proof. Let M be a $(0,1)$-matrix in which every row/column has at most one non zero entry. Take the left most 2×2-submatrix M_1 which is an augmented non singular matrices, i.e. has the form given in Figure 24. If this is a maximal augmented non singular matrix then this a block. Otherwise, this is not a maximal augmented non singular matrix then there is an other augmented non singular matrix M_2 such that $M_1 \cap M_2 \neq \emptyset$. This implies that $M_1 +_v M_2$ or $M_1 +_v M_2$ exists. On this way we get a maximal augmented non singular matrix. We consider as blocks the maximal augmented non singular $(k \times k)$- matrices. The remaining entries form 1×1 blocks with "0" entries.

Intuitively, we have the "1" entries in the plain, some areas are "density areas" of these entries, these generate a block which is a maximal augmented non singular matrix; the "isolated "1"-s are one-element blocks.

□

Remark. The given matrix M has a block system, where the blocks with more one entry are maximal augmented non singular matrices. To make the picture more spectacular we can color the blocks and we obtain a "patchwork" of blocks, i.e. of augmented non singular matrices.

10.1. **The three-dimensional case, hypermatrices.** We define the 3-dimensional hyper boxes and as partial operation we take the convex hull as "convex rectangular sub-hypetmatrices. Then we take the system of maximal boxes. This determines a patchwork of the given lattice.

Definition 10. *A* block *of a 3D hypermatrix $M = [a_{i,j,k}]$, $1 \leq i,j \leq n$ is a 3D sub-hypermatrix in the form $[a_{i,j,k}]$, $i \in \{s, s+1, s+2, ..., s+l\}$, $j \in \{t, t+1, t+2, ..., t+l\}$ $j \in \{k, k+1, k+2, ..., k+l\}$for some s, t, k, l.*

See in Figure 22, the rows/columns of the block are consecutive rows/columns of the given matrix (geometrically it is a convex rectangular). A *block hypermatrix* is a system of blocks such that the (set theoretical) meet of two blocks does not contain any entry and every entry is in a block (let us remark that this definition is not the usual definition of block marices).

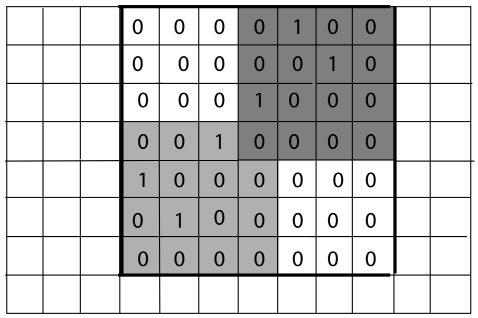

FIGURE 30. Horizontal sum of blocks, $+_h$

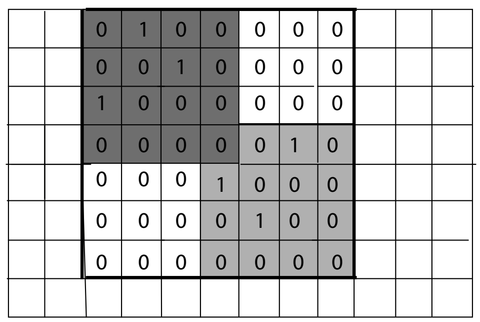

FIGURE 31. Vertical sum of blocks, $+_v$

11. The structure theorem (conjecture)

The following theorem was proved for two-dimensional lattices by G. Cédli and E. T. Schmidt in [4]:

Theorem 7. *Every finite two-dimensional semimodular lattice R is the patchwork of patch lattices.*

Conjecture 2. *Every finite semimodular lattice R is the patchwork of patch lattices.*

Problem 5. *Establish the connection between the patchwork of block matrices and patchwork of 2D semimodular lattices. Prove that Theore 7 follows from Lemma 7.*

Remarks to the 2D case. Let K be a two-dimensional semimodular lattice, see Figure 31. We my assume that K is rectangular [12]. Take the source $\mathcal{S}$ and the matrix $\mathbf{M}_K$ derived from the lower grid $(G, \mathcal{S})$. This is a (0, 1)-matrix and every row/column contains at most one non zero entry. By Lemma 4 $\mathbf{M}_K$ has blocks wich are matrices of patch lattices and some 1×1-matrices (with "0" entries). We get a patch system of K (Figure 31). The "1" entries give the source, which determine a cover-preserving join congruence Θ. Applay this to the blocks we get the patch lattices (in the example the L_i-s), see in Figure 32. All "unit squers" of the matrix which do'nt below to a block we put 0 entrie. Such a "unit squer" vanish if we factotize or gives a 1×1 block.

In the 2D case the form the surce which is a s-independent subset and therefore the restriction of a s-inependent to an interval is again s-indpendent subset, i.e. the blocks determine patch lattice.

In Figure 32 we see 3 semimodular lattices $L_1 = N_7, L_2 N_7$ and $L_3 = {C_2}^2$ patched together.

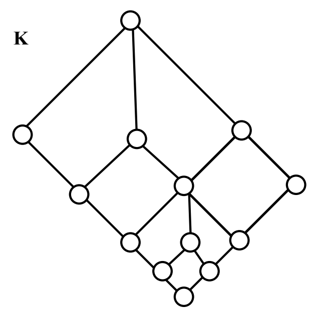

FIGURE 32. The lattice K

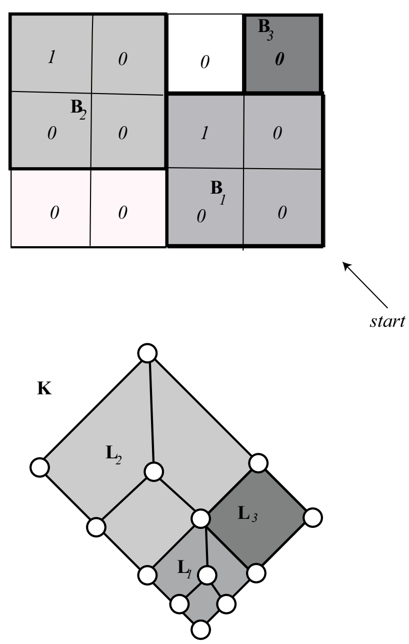

FIGURE 33. Patching of matrices and K

Remarks to the 3D case.

By 3-dimensional lattices we use hypermatrices. We begin again with a 3-dimensional semimodular lattice K and then we take its cube-hypermatrix $\mathbf{M}_K$. Every row/column of $\mathbf{M}_K$ contains at most one non-zero entry (which is an "1" and form the poset S, the source elements. Every block must satisfy the condition (3D).

We define in the 3D case the patch lattice as a nested Boolean lattice.

Problem 6. *Characterize the hypermatrices of 3D patch lattices.*

The difficulties makes the condition (3D), see Figure 33. L has a source $\mathcal{S} = \{s_1, s_2, s_3\}$, it satisfis (3D). The lattice L is the Hall-Dilworth gluing of two lattices (cubes), A and B. The restriction of $\mathcal{S}$ to A is not a soure, (DC) is not satisfied, we need a new source element s_4 and delete s_3. The s_4 is the projection of s_3 to A (s_3 and s_4 are not s-independent).

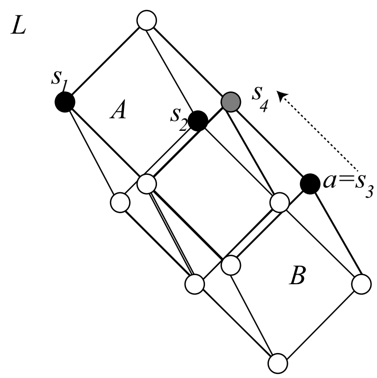

Figure 34. Hall-Dilworth and the condition (3D)

12. Appendix: modular lattices.

Finally we give an example of a patchwork of modular lattices. The Fano plane is a seven-dimensional modular lattice, $\mathbf{dim}(F) = 7$, $(\mathbf{Dim}(F) = \mathbf{dim}_{\mathbf{KO}}(F) = 3)$.

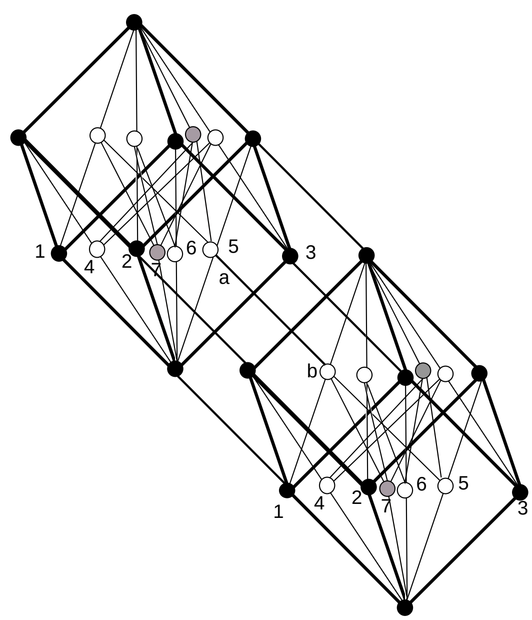

Figure 35. Patchwork of Fano planes and $M_3 \times \mathcal{C}_2$

References

[1] G. Czédli, E. T. Schmidt, *How to derive finite semimodular lattices from distributive lattices?*, Acta Math. Acad. Sci. Hungar.,**121** (2008), 277–282.

[2] G. Czédli, E. T. Schmidt, *Intersections of composition series in groups and slim semimodular lattices by permutations,* submitted

[3] G. Czédli and E. T. Schmidt, *Slim semimodular lattices. I. Visual approach*, Order, **29** (2012), —(DOI: 10.1007/s1083-011-9215-3)

[4] G. Czédli, E. T. Schmidt, *Slim semimodular lattices. II. A description by patchwork systems*—, manuscript

[5] G. Czédli and E. T. Schmidt, *Compositions series in groups and the structure of slim semimodular*, Advances in Mathematics, (May 14, 2011)

[6] G. Czédli, *The matrix of slim semimodolar lattices*, Order, **29** (2012), 85–103.

[7] G. Czédli, Ozsvárt, Udvari *How many ways can two composition series intersect?*, Discrete Mathematics, **312** (2010), 3523–3536

[8] G. Czédli and E. T. Schmidt, *Finite distributive lattices are congruence lattices of almost-geometric lattices*, Algebra Universalis, **65** (2011), 91-108. (DOI: 10.1007/s00012-011-0119-2)

[9] P. H. Edelman, R. E. Jamison, *The theory of convex geometries*, Geom. Dedicata, **10** (1985), 247–271.

[10] E. Fried, G. Grätzer and E. T. Schmidt, *Multipasting of lattices*, Algebra Universalis **30** (1993), 241–261.

[11] G. Grätzer, E. Knapp, *Notes on planar semimodular lattices. I. Constructions,* Acta Sci. Math. (Szeged) **73** (2007), 445–462.

[12] G. Grätzer, E. Knapp, *Notes on planar semimodular lattices. III, Congruences of rectangular lattices,* Acta Sci. Math. (Szeged) **74** (2009), 29–48.

[13] C. Herrmann, *S-verklebte Summen von Verbänden.* Math. Z. **130** (1973), 255-274.

[14] E. T. Schmidt, *Rectangular hulls of semimodular lattices,* manuscript.

[15] E. T. Schmidt, *A new look at the semimodular lattices. A collection of ideas and conjectures*, manuscript.

[16] E. T. Schmidt, *Play with matrices get a structure theorem for semimodular lattices,* manuscript.

[17] E. T. Schmidt, *A structure theorem of semimodular lattices: the patchwork representation,* manuscript.

[18] E. T. Schmidt, *Diamond-free 3-dimensional semimodular lattices* manuscript.

[19] E. T. Schmidt, *Rectangular lattices as geometric shapes* manuscript.

[20] M. Stern, *Semimodular Lattices. Theory and Applications.* Encyclopedia of Mathematics and its Applications, **73**. Cambridge University Press (1999)

Mathematical Institute of the Budapest University of Technology and Economics, Műegyetem rkp. 3, H-1521 Budapest, Hungary

E-mail address, E. T. Schmidt: `schmidt@math.bme.hu`

URL: `http://www.math.bme.hu/~schmidt/`