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Cover-preserving embeddings of finite length semimodular lattices into simple semimodular lattices

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Abstract. We prove that every semimodular lattice of finite length has a cover-preserving embedding as a filter into a simple semimodular lattice.

G. Grätzer and E. W. Kiss [3] proved the following deep theorem: every finite semimodular lattice L has a cover-preserving embedding into a finite simple geometric lattice. For semimodular lattices of finite length see G. Czédli and E. T. Schmidt [1]. By the length, $\ell(L)$, of a lattice L we mean the number $\sup\{n \mid L \text{ has an } (n+1)- \text{ element chain}\}$.

If we drop 'geometric', G. Grätzer and T. Wares [5] proved that every finite semimodular lattice has a cover-preserving embedding into a simple semimodular lattice. Their proof is relatively short and uses the one-element extension introduced in [4]. In the present note we give a very short proof of this result.

Theorem. *Every semimodular lattice L of finite length has an embedding φ into a simple semimodular lattice $\overline{L}$ which satisfies the following conditions:*

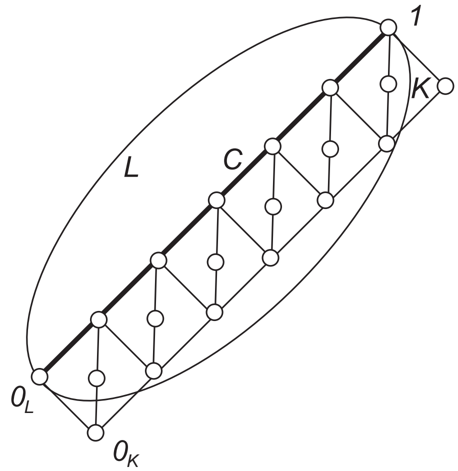
(1) *φ is cover-preserving,*
(2) *$L\varphi$ is a filter of $\overline{L}$,*
(3) *$\ell(\overline{L}) = \ell(L) + 1$ and $|\overline{L}| = |L| + 2\ell(L) + 1$.*

Proof. Let L be a semimodular lattice of finite length. Take a maximal chain C of L. Consider $\{0,1\} \times C$ and add new elements to the covering squares making them into copies of the lattice M_3. We get a lattice K that contains a filter $C' = \{(1,c); c \in C\}$ isomorphic to C under the isomorphism $\psi : C \to C'$, which sends every $c \in C$ to $(1,c)$. Consider the attachment $\overline{L}$ of the lattice K to the lattice L over C by identifying C with C' along ψ, in the sense of G. Grätzer and D. Kelly [2] (see the Figure). So we have $\overline{L} = L \cup K$ with $L \cap K = C$ and the order

$$a \leq b = \begin{cases} a \leq_K b & \text{if } a, b \in K; \\ a \leq_L b & \text{if } a, b \in L; \\ a \leq_K x\psi, \ x \leq_L b & \text{if } a \in K, \ b \in L, \text{ for some } x \in C. \end{cases}$$

The poset $\overline{L}$ is obviously a lattice and L is a filter, which proves (2). To show that $\overline{L}$ is semimodular, we have to verify the condition: $a \prec b$ implies $a \vee d \preceq b \vee d$.

Key words and phrases: Lattice, semimodular, finite length, simple, embedding.
This research was supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. K 77432.



There is only one nontrivial case: $a \in \overline{L} - L, d \in \overline{L} - K$, then $b \in K$. Take an arbitrary $c \in C$ with $c \leq d$. Then $a \vee c \preceq b \vee c$ in $C' \subset K$ and therefore $a \vee d = a \vee (c \vee d) = (a \vee c) \vee d \preceq (b \vee c) \vee d = b \vee d$.

K is simple, consequently, if two distinct elements of C in $\overline{L}$ are congruent under a congruence Θ, then $\Theta = \iota$ (the unit congruence). If we have a covering pair $a \prec b$ in L, then b/a is prime-projective to a prime quotient of C (see the strengthened version of the Jordan-Hölder theorem, for example, see J. B. Nation [6], Theorem 9.7), which proves that $\overline{L}$ is simple. Conditions (1) and (3) are obviously satisfied.

□

References

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