CONGRUENCE LATTICES AND COVER PRESERVING EMBEDDINGS OF FINITE LENGTH SEMIMODULAR LATTICES. I

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Abstract. Let $\mathcal{K}$ denote the class of finite length semimodular lattices that have congruence-determining chain ideals. Assume that $L \in \mathcal{K}$ and D is a $(0,1)$-sublattice of $\operatorname{Con} L$. We prove the existence of an $\overline{L} \in \mathcal{K}$ such that L is a filter of $\overline{L}$ and the restriction mapping from $\operatorname{Con} \overline{L}$ to $\operatorname{Con} L$ is an isomorphism from $\operatorname{Con} \overline{L}$ onto D. The particular case $D = \{0,1\}$, not only for $L \in \mathcal{K}$, has intensively been studied by several papers, including [4], [5], [2] and [7].

1. Introduction

Let B be a sublattice of a lattice A. If $0_A, 1_A \in B$, then B is said to be a $(0,1)$-sublattice. If every congruence of A is determined by its restriction to B, then B is called a *congruence-determining* sublattice of A. Ideals that are chains will be called *chain ideals*. Consider the class $\mathcal{K} = \{L : L$ is a finite length semimodular lattice that has a congruence-determining chain ideal$\}$.

Theorem 1. *Let $L \in \mathcal{K}$, and let D be a $(0,1)$-sublattice of $\operatorname{Con} L$. Then there exists an $\overline{L} \in \mathcal{K}$ such that the restriction mapping $\rho : \operatorname{Con} \overline{L} \to \operatorname{Con} L$, $\theta \mapsto \theta\rceil_L$, is actually a $(0,1)$-lattice isomorphism $\operatorname{Con} \overline{L} \to D$; in particular, $\operatorname{Con} \overline{L} \cong D$.*

Remarks. Theorem 1 asserts that $(0,1)$-lattice embeddings $D \to \operatorname{Con} L$ (D is a finite distributive lattice and $L \in \mathcal{K}$) can be represented by appropriate restriction mappings. Clearly, L is a cover-preserving sublattice and also a congruence-determining sublattice of $\overline{L}$. The present proof is an improvement of [7]. The particular case $D = \{0,1\} \subseteq \operatorname{Con} L$ (intensively studied in several papers, including [4], [5], [2] and [7]) means that we extend L to a *simple* lattice $\overline{L}$.

Let $\mathfrak{p}$ be a prime interval of L, and let $\operatorname{Prime}(L)$ denote the set of all prime intervals. Let $\operatorname{con}(\mathfrak{p})$ denote the smallest congruence under which $\mathfrak{p}$ collapses. The congruences of the form $\operatorname{con}(\mathfrak{p})$ are exactly the join-irreducible elements of $\operatorname{Con} L$. By a *colored lattice* we will mean a lattice of finite length of whose prime intervals are labeled so that if the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ are of the same color, then $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{q})$. The colors are the join-irreducible elements of $\operatorname{Con} L$, i.e., elements of $\operatorname{J}(\operatorname{Con} L)$. (Coloring is a useful tool to understand the construction.)

Let $\mathfrak{p} \Rightarrow \mathfrak{q}$ mean that $\mathfrak{p}$ is congruence-projective onto $\mathfrak{q}$. This defines a preordering (a reflexive and transitive relation) on $\operatorname{Prime}(L)$. $\mathfrak{p} \Leftrightarrow \mathfrak{q}$ denotes that $\mathfrak{p} \Rightarrow \mathfrak{q}$ and

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$\mathfrak{q} \Rightarrow \mathfrak{p}$. The equivalence classes under $\Leftrightarrow$ form an order isomorphic to $\mathrm{J}(\mathrm{Con}\, L)$. Note that $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$ in $\mathrm{J}(\mathrm{Con}\, L)$ is equivalent to $\mathfrak{p} \Rightarrow \mathfrak{q}$.

Let $L \in \mathcal{K}$. We fix a congruence-determining chain ideal $C = \{0 = c_0 < c_1 < ... < c_n\}$ of L. The prime intervals of C are denoted by $\mathfrak{p}_i = [c_{i-1}, c_i]$ $(i = 1, 2, ..., n)$. By the choice of C, each prime interval of L is equivalent to a prime interval of C. Hence $\mathrm{Con}\, L$ is finite.

2. A semimodular lattice K

Let D be a $(0,1)$-sublattice of $\mathrm{Con}\, L$. For $\mathfrak{p} \in \mathrm{Prime}(L)$, denote $\overline{\mathrm{con}}(\mathfrak{p})$ the smallest congruence with the properties: $\overline{\mathrm{con}}(\mathfrak{p}) \in D$ and $\mathrm{con}(\mathfrak{p}) \leq \overline{\mathrm{con}}(\mathfrak{p})$. Clearly, for each $\mathfrak{p}_i \in \mathrm{Prime}(C)$, $\overline{\mathrm{con}}(\mathfrak{p}_i)$ is the join of some $\mathrm{con}(\mathfrak{p}_j)$, $\mathfrak{p}_j \in \mathrm{Prime}(C)$.

This means that D is determined by the set $\mathfrak{R}_D$ of all pairs $(\mathfrak{p}_\mathfrak{i}, \mathfrak{p}_\mathfrak{j})$ of prime intervals of C satisfying $\overline{\mathrm{con}}(\mathfrak{p}_\mathfrak{i}) \geq \mathrm{con}(\mathfrak{p}_j)$. We may assume that $\mathfrak{p}_i \not\Rightarrow \mathfrak{p}_j$ in L for these pairs. We are going to define a semimodular lattice K which is colored by the elements of $\mathrm{J}(\mathrm{Con}\, L)$. In K, $\mathfrak{p}_i \Rightarrow \mathfrak{p}_j$ for all $(\mathfrak{p}_i, \mathfrak{p}_j) \in \mathfrak{R}_D$ but no other implications are satisfied. We will obtain $\overline{L}$ as a Hall-Dilworth gluing of L and K. Note that the join-irreducible elements in D are the $\overline{\mathrm{con}}(\mathfrak{p}_i)$, but $\overline{\mathrm{con}}(\mathfrak{p}_i) = \overline{\mathrm{con}}(\mathfrak{p}_j)$ may happen even if $\mathfrak{p}_i$ is not congruence-equivalent with $\mathfrak{p}_j$ in L. The planar semimodular lattice $K = K(\mathfrak{R}_D)$ will have the following properties:

(1) K has a congruence-determining chain filter $C' = \{d_0 < d_1 < ... < d_n\}$; it is isomorphic to C.
(2) K has a congruence-determining chain ideal.
(3) Denoting the prime interval $[d_{s-1}, d_s]$ by $\mathfrak{q}_s$ $(1 \leq s \leq n)$, $\mathrm{con}_K(\mathfrak{q}_i) \geq \mathrm{con}_K(\mathfrak{q}_j)$ iff $(\mathfrak{p}_i, \mathfrak{p}_j) \in \mathfrak{R}_D$.

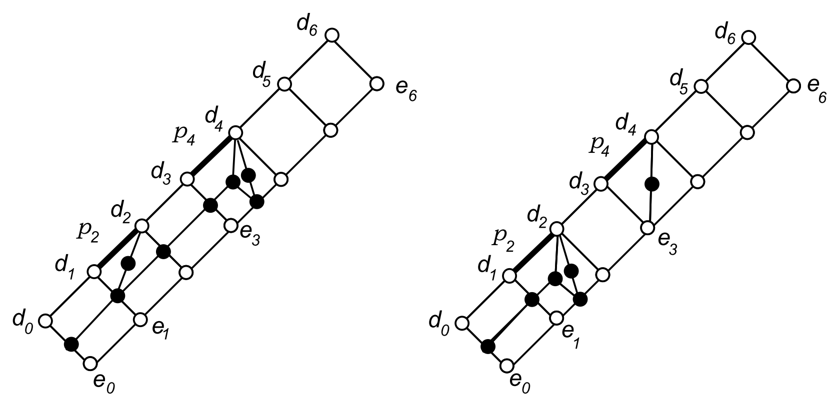

Figure 1. The lattice T_1 in the $n = 6$ case

Step 1: Consider an arbitrary $(\mathfrak{p}_i, \mathfrak{p}_j) \in \mathfrak{R}_D$. Take the direct product $\{0, 1\} \times \{0, 1, ..., n\}$ of two chains. Then, as depicted in Figure 1, we define T_1 by adding the black-filled elements to this direct product; depending on whether $i > j$ (see $i = 4$ and $j = 2$ on the left side), or $i < j$ (see $i = 2$ and $j = 4$ on the right side). This T_1 is a semimodular lattice with a filter $C' = \{d_k = (1, k) : k \in \{0, 1, ..., n\}\}$, and $\psi\colon C \to C'$, $c_k \mapsto d_k$ is an isomorphism. Hence, by identifying $\mathfrak{p}_s$ with $[d_{s-1}, d_s]$, we can extend the coloring of L to C' and, since C' is congruence-determining in T_1, to the whole T_1. Clearly, the desired $\mathfrak{q}_i \Rightarrow \mathfrak{q}_j$ holds in T_1, but $\mathfrak{q}_k \not\Rightarrow \mathfrak{q}_l$ if $(i,j) \neq (k,l)$. Furthermore, T_1 has a congruence-determining chain ideal, $(e_6]$ in Figure 1, which consists of $n+2$ elements.

Step 2: Consider an other element $(\mathfrak{p}_k, \mathfrak{p}_l)$ of $\mathfrak{R}_D$, and take the direct product $\{0,1\} \times \{0,1,...,n+1\}$ of two chains. Add the black filled elements to two covering squares as in Step 1 to accomplish the relation $\mathfrak{q}_k \Rightarrow \mathfrak{q}_l$. We get a semimodular lattice T_2. T_1 has an ideal which is an $n+2$-element chain. T_2 has a dual ideal which is an $n+2$-element chain. Glue these lattices together via these two chains. We obtain $K_2 = T_1 \bigcup T_2$, (and $K_1 = T_1$), and we extend the coloring of T_1 to K_2.

Step 3: We continue this procedure similarly for the elements of $\mathfrak{R}_D$. This way we get a finite sequence of lattices $K_1, K_2, K_3, \ldots$. Finally, $K = \cup K_i$. This is obviously a planar semimodular lattice that satisfies conditions (1), (2) and (3).

3. The proof of Theorem 1

We have already fixed a congruence-determining chain ideal C of $L \in \mathcal{K}$. This C and D determine $\mathfrak{R}_D \subseteq \mathrm{Prime}(L) \times \mathrm{Prime}(L)$ and a planar semimodular lattice K that satisfies (1), (2) and (3). Let $\overline{L}$ be the Hall-Dilworth gluing of K and L that identifies C with C', see Figure 2. So $\overline{L} = L \cup K$, $L \cap K = C$ and, for $a, b \in \overline{L}$,

$$a \leq b \ \text{ means that } \begin{cases} a \leq_K b & \text{if } a, b \in K; \\ a \leq_L b & \text{if } a, b \in L; \\ a \leq_K c, \ c \leq_L b & \text{if } a \in K, \ b \in L \text{ for some } c \in C. \end{cases}$$

Clearly, $\overline{L}$ is of finite length. Its semimodularity follows in a straightforward way. Namely, let $a \prec b$ in $\overline{L}$. Then $a, b \in K$ or $a, b \in L$; we give the details only for the case $a, b \in L$. Then we can assume that $b \in K$ since otherwise $a \vee d \preceq b \vee d$ is evident. There is a $c \in C$ with $a \leq c \leq a \vee d$. Therefore, $a \vee d = a \vee c \preceq b \vee c = b \vee d$ by the semimodularity of L. Hence $\overline{L}$ is semimodular.

Take two prime intervals $\mathfrak{p}_\mathfrak{i}, \mathfrak{p}_\mathfrak{j}$ of $C = L \cap K$. Then $\mathfrak{p}_i \Rightarrow \mathfrak{p}_j$ in $\overline{L}$ iff either $\mathfrak{p}_i \Rightarrow \mathfrak{p}_j$ in the original lattice L, or $\mathfrak{p}_i \Rightarrow \mathfrak{p}_j$ in K. By the construction of K the second possibility occurs iff $(\mathfrak{p}_i, \mathfrak{p}_j) \in \mathfrak{R}_D$. This determines D, i.e. a congruence relation Φ of L extends to $\overline{L}$ iff $\Phi \in D$, and every congruence $\Theta \in D$ has exactly one extension $\overline{\Theta}$ to $\overline{L}$. This implies that $\mathrm{Con}\,\overline{L} \cong D$ and that the restriction mapping $\rho\colon \mathrm{Con}\,\overline{L} \to \mathrm{Con}\,L$ is a $\mathrm{Con}\,\overline{L} \to \mathrm{Con}\,D$ isomorphism.

Finally, the congruence-determining chain ideal of K is a congruence-determining chain ideal of in $\overline{L}$. This proves the theorem.

4. Concluding comments

Rectangular lattices were introduced by G. Grätzer and E. Knapp [6] as follows. A *left corner* (resp. *right corner*) of a planar lattice S is a double-irreducible element of $S \setminus \{0,1\}$ on the left (resp., right) boundary of S. A *rectangular* lattice R is a planar semimodular lattice which has exactly one left corner, u_l, and exactly one right corner, u_r, and they are complementary, that is, $u_l \vee u_r = 1$ and $u_l \wedge u_r = 0$. For example, K from the previous section is clearly rectangular. The strength of Theorem 1 is demonstrated by the following

Corollary 1 (Grätzer and Knapp [6]). *Each finite distributive lattice D is isomorphic to the congruence lattice of a rectangular lattice $\overline{L}$.*

Proof. Let L be a chain of length $n = \mathrm{J}(D)$, and apply Theorem 1. □

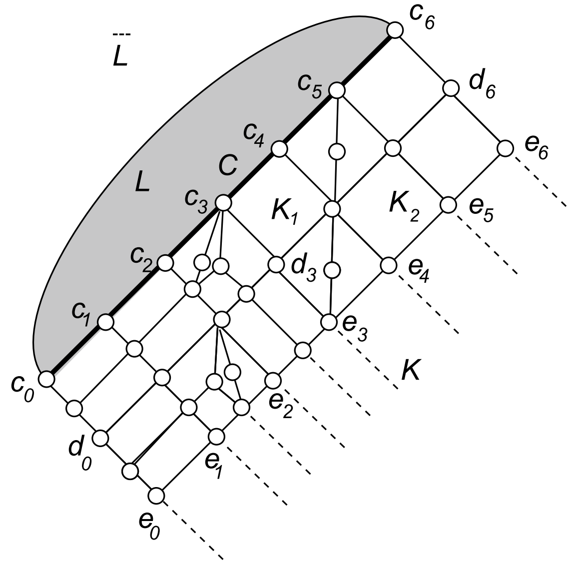

FIGURE 2. The lattice $\overline{L}$.

Added in proof, March 15, 2011: There are many rectangular lattices, like the four-element one, that do not belong to $\mathcal{K}$. However, as it was pointed out in [1], the second theorem of [3] easily allows us to extend Theorem 1 to the class of all rectangular lattices instead of $\mathcal{K}$. Further extensions will be given in a forthcoming paper.

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