ON A PROBLEM OF L. FUCHS CONCERNING UNIVERSAL SUBGROUPS AND UNIVERSAL HOMOMORPHIC IMAGES OF ABELIAN GROUPS

BY

G. GRÄTZER AND E. T. SCHMIDT (Budapest)

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In this note our aim is to prove the following

Theorem. A necessary and sufficient condition for an abelian group $G$ to contain a universal subgroup $Z$ such that $G/Z$ is a universal homomorphic image is

(a) for $p$-groups: $r(p^iG) = O$ or infinite, for every integer $i$;
(b) for torsion groups: every $p$-component of $G$ fulfills (a);
(c) for groups with torsion free rank $r(>0)$: $r$ is infinite.

This theorem solves completely a problem of L. Fuchs ([1], p. 352, Problem 85).

For convenience sake we call a universal subgroup $Z$ perfect if $G/Z$ is a universal homomorphic image.

Remark 1. Condition (a) means the following: $G$ is either bounded and $p^{k-1}G$ is infinite, where $k$ is the smallest integer with $p^kG = 0$, or $G$ is unbounded and the final rank of $G$ (fin $r(G) = \min_i r(p^iG)$) is infinite.

Remark 2. Comparing our theorem with a theorem of L. Fuchs we find that the existence of universal subgroups implies in almost all cases the existence of a perfect universal subgroup $Z$. The exceptions are:

(a) $G$ is a bounded $p$-group such that $p^{k-1}G$ is finite but not $0$; (b) $G$ is a torsion group every $p$-component of which fulfills (a) or (a') and at least one $p$-component fulfills (a); (c) the torsion free rank of $G$ is a natural integer $r$ and $G = T + \sum \mathbb{C}(\infty)$, where $T$ is a torsion group satisfying (b) or (b').

For the notions and notations we refer to [1].

We need the trivial

Lemma. If $Z_1$ is a universal subgroup of $G$ and $U_1$ a universal homomorphic image and if we have for a subgroup $Z$ of $G$

1) A subgroup $Z$ of $G$ is a universal subgroup if every subgroup of $G$ is isomorphic to a homomorphic image of $Z$ (see [1], p. 341, or [2]).
2) A homomorphic image $U$ of $G$ is called universal if every homomorphic image of $G$ is isomorphic to some subgroup of $U$ (see [1] p. 336, or [2]).
3) See [1], p. 343, or [2].
1°. \( Z_1 \) is a homomorphic image of \( Z \);
2°. \( U_1 \) is isomorphic to a subgroup of \( G/Z \),
then \( Z \) is a universal subgroup and \( G/Z \) is a universal homomorphic image of \( G \).

**Proof of the Theorem.**

**Case (a). Necessity.** From the theorem of Fuchs, mentioned in remark 2, we know that \( \text{fin} \, r(G) = 0 \) or infinite. Hence we need only consider the case that \( r(p^k G) = 0 \), \( 0 < r(p^k G) < \infty \). Then \( Z \subseteq G, Z \sim G \)
imply \( r(p^{k-1} Z) = r(p^{k-1} G) \), thus \( p^{k-1} Z = p^{k-1} G \) and it is impossible that \( G \)
is isomorphic to a subgroup of \( G/Z \).

**Sufficiency.** First, let \( G \) be a bounded \( p \)-group with the minimal bound \( p^k \). Then \( G = G_1 + \cdots + G_k \), where \( G_i = \sum C(p^i) \) and \( r(G_k) = r(p^k G) \); thus the condition implies that \( G_k \) is infinite. We put \( G_i = G_i' + G_i'' \), where
\( G_i' = 0 \) or \( G_i'' \cong G_i' \) according as \( G_i \) is finite or not. Then \( Z = \sum_{i=1}^{k} G_i' \)
is a perfect universal subgroup, for we may choose in the Lemma \( G \cong Z_1 \cong U_1 \),
as \( Z \sim G \) (in fact, \( Z \cong G \) ); that \( G \) is isomorphic to a subgroup of \( G/Z \)
is trivial.

Secondly, if \( G \) is unbounded, \( \text{fin} \, r(G) > \aleph_0 \) follows from the condition.
Then we may decompose 4° \( G = G_1 + G_2 \), where \( G_1 \) is a bounded group satisfying (a) and \( \text{fin} \, r(G) = r(G_2) = m \). It follows that \( G_2 \) contains a subgroup \( F \) isomorphic to the free \( p \)-group 5° \( F_p(m) \); let \( B \) be a lower basic subgroup of \( F \). We define \( Z = Z' + B \), where \( Z' \) is a perfect universal subgroup of \( G_1 \).

**The case (b) is trivial.**

**Case (c). Necessity.** If the torsion free rank \( r(\cdot > 0) \) is finite, then \( Z \sim G \),
\( G \subseteq Z \) imply \( r = r_0(G) = r_0(Z) \) and thus \( r_0(G/Z) = 0 \), contradicting the fact that \( G \) is isomorphic to a subgroup of \( G/Z \).

**Sufficiency.** Let \( r \) be infinite. We may decompose 7° \( G = G_1 + G_2 \) such
that \( G_1 \) is a torsion group with bounded \( p \)-components satisfying (b),
thus having a perfect universal subgroup \( Z' \) and \( G_2 \) contains subgroups
\( H \cong F(r) \) and \( H_i \cong F_{p_i}(m_i) \) where \( m_i \) denotes the final rank of the \( p_i \)-component of the maximal torsion subgroup of \( G_i \)
while
\[
\{H, F_1, F_2, \ldots\} \cong H + \sum_i F_i.
\]
We define \( Z = Z' + K + B \), where \( K \subseteq H \), \( H/K \cong \sum_r R + \sum_{m_i \leq t} \sum C(p^\infty) \) where
the summation is for all \( i \) with \( m_i < t = \max (r, m_i) \), further \( B \) is a lower

4°) This follows easily from Theorem 31.5 of [1].
5°) See [1], p. 39.
6°) See [1], p. 165 and Theorem 31.4.
7°) Lemma 87.2 of [1].
basic subgroup of \( \sum_{m_i \gg M_0} H_i \). We may choose \(^8\) \( U_1 = G_1 + \sum_{t} R + \sum_{t} \sum_{t_i} C(p_i^{r_i}) \) and \( Z_1 = G_1 + F(t) + \sum_{m_i \gg M_0} F_R(m_i) \).

Now \( Z \cong Z_1 \) and that \( U_1 \) is isomorphic to a subgroup of \( G/Z \) is clear from \( \sum_{m_i \gg M_0} H_i \cong \sum_{m_i \gg M_0} C(p_i^{r_i}) \). Thus the proof of the theorem is complete.

\(^8\) See [1], p. 338 and p. 342.

**LITERATURE**