On $n$-permutable equational classes

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The product $\Theta \circ \Phi$ of two congruences $\Theta, \Phi$ of an algebra $A$ is defined by the following rule: $a \equiv b(\Theta \circ \Phi)$ if and only if $c \in A$ exists such that $a \equiv c(\Theta)$ and $c \equiv b(\Phi)$. Two congruences $\Theta_1$ and $\Theta_2$ are $n$-permutable if and only if $\Theta_1 \circ \Theta_2 \circ \Theta_1 \circ \Theta_2 \circ \cdots = \Theta_2 \circ \Theta_1 \circ \Theta_2 \circ \Theta_1 \circ \cdots$, where on both sides there are $n$ factors. An algebra $A$ is $n$-permutable if every two congruences in $A$ are $n$-permutable. We define an equational class to be $n$-permutable if every algebra of this class is $n$-permutable. It is well known, that an $n$-permutable equational class is $(n+1)$-permutable. In [1] G. Grätzer asks for examples of equational classes which show that $n$-permutability and $(n+1)$-permutability are not equivalent\(^1\). In this note we give an example with this property.

**Theorem.** For every natural number $n \geq 2$ there exists an $(n+1)$-permutable equational class $\mathcal{K}_n$ which is not $n$-permutable.

**Proof.** Let $n$ be a natural number. An $n$-Boolean algebra

$$\mathcal{B}=\{B; \lor, \land, f_1(x), \ldots, f_n(x), o_0, o_1, \ldots, o_n\}$$

is an algebra with two binary operations $\lor, \land$, $n$ unary operations $f_1(x), \ldots, f_n(x)$ and $n+1$ nullary operations $o_0, o_1, \ldots, o_n$, such that the following conditions are satisfied:

1. $(B; \lor, \land)$ is a distributive lattice;
2. $x \lor o_0 = o_n, x \lor o_0 = x$ for all $x \in B$;
3. $[(x \lor o_{i-1}) \land o_i] \lor f_i(x) = o_i, [(x \lor o_{i-1}) \land o_i] \land f_i(x) = o_{i-1}$.

The class of all $n$-Boolean algebras is denoted by $\mathcal{K}_n$. If $o_{i-1} \equiv x \equiv o_i$ then $f_i(x)$ is the relative complement from $x$ in $[o_{i-1}, o_i]$, i.e. this interval is a Boolean lattice. A 1-Boolean algebra is a Boolean algebra. A finite chain $\mathcal{C}_n$ of $n+1$ elements is

\(^1\) For $n=2$ A. Mitschke [2] has solved this problem.
an $n$-Boolean algebra, if we take its elements as nullary operations: $a_0 = a_1 = a_2 = \ldots = a_n$ ($o_i \in \mathcal{G}_n$), and $f_i(x) = a_i$ if $x = a_i$, $f_i(x) = a_{i-1}$ if $x \equiv a_i$. The congruences of $\mathcal{G}_n$ are the lattice-congruences, i.e., $\mathcal{G}_n$ is not $n$-permutable. This shows that $\mathcal{N}_n$ is not $n$-permutable.

Let $B$ denote an arbitrary $n$-Boolean algebra and $x, y \in B, x \geq y$. Set $a_i = (o_i \wedge x) \vee y$. (Then is $a_0 = y, a_n = x$.) If $\Theta_1$ and $\Theta_2$ are arbitrary congruences from $B$, such that $x \equiv y (\Theta_1 \vee \Theta_2)$, then $a_{i-1} \equiv a_i (\Theta_1 \vee \Theta_2)$ ($i = 1, 2, \ldots, n$). The interval $[a_{i-1}, a_i]$ is projective to a subinterval of $[o_{i-1}, o_i]$, i.e., $[a_{i-1}, a_i]$ is a Boolean lattice. Every Boolean lattice is 2-permutable and so for every $i$ ($i = 1, 2, \ldots, n$) there exists a $t_i \in [a_{i-1}, a_i]$ such that

$$a_{i-1} \equiv t_i (\Theta_1) \text{ if } i \text{ odd, } a_{i-1} \equiv t_i (\Theta_2) \text{ if } i \text{ even, } a_i \equiv t_i (\Theta_1) \text{ if } i \text{ even, } a_i \equiv t_i (\Theta_2) \text{ if } i \text{ odd.}$$

We have therefore between $x, y$ a chain $y_0 = a_0 = y, y_1 = t_1, y_2 = t_2, \ldots, y_n = x = a_n$ with $n + 1$ elements, such that $y_{i-1} \equiv y_i (\Theta_1)$ if $i$ even and $y_{i-1} \equiv y_i (\Theta_2)$ if $i$ odd. $\mathcal{N}_n$ is therefore $(n+1)$-permutable.

**Remark.** An equational class is $(n+1)$-permutable if and only if there exists $(n+2)$-ary algebraic operations $p_0, \ldots, p_{n+1}$ satisfying the following identities (see [3]):

$$p_0(x_0, \ldots, x_{n+1}) = x_0, \quad p_{i-1}(x_0, x_0, x_2, x_2, \ldots) = p_i(x_0, x_0, x_2, x_2, \ldots) \quad (i \text{ even}),$$

$$p_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) = p_i(x_0, x_1, x_1, x_3, x_3, \ldots) \quad (i \text{ odd}),$$

$$p_{n+1}(x_0, \ldots, x_{n+1}) = x_{n+1}.$$

A. MITSCHKE and H. WERNER have considered for the class $\mathcal{N}_n$ the algebraic operations:

$$p_i(x_0, x_1, \ldots, x_{n+1}) = (x_i \wedge f_{n+1-i}(x_{i+1}) \vee x_{i+2}) \vee (x_{i+2} \wedge (f_i(x_{i+1}) \vee x_i))$$

which show that $\mathcal{N}_n$ is $(n+1)$-permutable.

**Bibliography**


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