EVERY FINITE DISTRIBUTIVE LATTICE IS THE
CONGRUENCE LATTICE OF SOME MODULAR LATTICE

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1. Introduction

The purpose of this paper is to prove the theorem formulated in the title. The
notation used is that of Grätzer [2]. The unary algebraic functions play by the
description of congruence relations a very important role. Let \( p = p(x) \) be an unary
algebraic function on the modular lattice \( K \), and let \( a_0 \leq b_0 \) be two elements of \( K \);
it is easy to show that there exists a pair \( a, b \in K \), \( a_0 \leq a \leq b \leq b_0 \) such that the restriction
of \( p \) to \([a, b]\), \( p|_{[a, b]} \) is an isomorphism between \([a, b]\) and \([p(a_0), p(b_0)]\), i.e.,
these intervals are projective. Let now \( f : [a, b] \to [c, d] \) be an arbitrary isomorphism,
then we take \( f \) as a partial unary operation with the domain \([a, b]\). We called such a
partial operation a \( \ast \)-operation. The inverse \( f^{-1} \) of \( f \) is again a \( \ast \)-operation. If
there exists for \( f \) an unary algebraic function \( p \) on \( K \), such that \( f = p|_{[a, b]} \), then \( p \) is
called a realization of \( f \).

Now we consider a sublattice \( K_0 \) of the lattice \( K \) (in other words \( K \) is an exten-
sion of \( K_0 \)). Then \( K \) determines a system of projective intervals \([a_0, b_0]\), \([c_0, d_0]\)
\( (x \in \Omega) \) of \( K_0 \). Let \( p_0 \) denote the unary algebraic function which maps \([a_0, b_0]\) into
\([c_0, d_0]\). With the corresponding \( \ast \)-operation \( \hat{p}_0 = p_0|_{[a_0, b_0]} \) we get a partial algebra
\( K_0^\ast = \langle K_0; \lor, \land, \hat{p}_0 \mid x \in \Omega \rangle \). A congruence relation of \( K_0 \) has an extension to \( K \) iff
\( \Theta \) is a congruence relation of \( K_0^\ast \).

DEFINITION. Let \( K_0 \) be a lattice and \( f_x : [a_0, b_0] \to [c_0, d_0] \) be \( \ast \)-operations of
\( K_0 \). If there exist an extension \( K \) of \( K_0 \) such that the following conditions are satisfied:
(1) every \( f_x \) has a realization in \( K \);
(2) for every \( \Theta \in C(K_0^\ast) \) there exists exactly one congruence relation \( \overline{\Theta} \) of \( K \) such
that for \( a, b \in K_0, a \equiv b (\overline{\Theta}) \) iff \( a \equiv b (\Theta) \), then we say that \( K \) is a realization of \( K_0^\ast \).

For a realization \( K \) of \( K_0^\ast, C(K) \cong C(K_0^\ast) \) obviously holds, i.e., the congruence
lattices are isomorphic.

EXAMPLE. Let \( K_0 \) be the three element chain: \( 0 < a < 1 \). Then \([0, a] \) and \([a, 1] \)
are isomorphic, so we have \( \ast \)-operations \( f : [a, 1] \to [0, a] \) and \( f^{-1} \). A realization of
\( \langle K_0; \lor, \land, f, f^{-1} \rangle \) is the following lattice, where \( p(x) = (((a \lor b) \land c) \lor d) \land a \)
realizes \( f \):

Presented by G. Grätzer. Received May 23, 1972. Accepted for publication in final form November 14,
We ask: does there exist for a modular lattice $K_0$ and *-operations a realization which is modular too? We prove, if $K_0$ is a chain then there always exists such a realization. In the second paragraph we give for a finite distributive lattice $L$ a chain $K_0$ and *-operations $f_i$ ($i=1, 2, ...$) with the property $L \cong C(K_0^*)$, where $K_0^* = \langle K_0; \lor, \land, f_i \rangle$. In the third paragraph there are proved two lemmas, and in the last paragraph the construction of $K$ is given, which realizes $K_0^*$.

2. A partial algebra

Let $Q$ be the chain of all rational numbers $r$ with $0 \leq r \leq 1$. Two non-trivial intervals $[a, b]$ and $[c, d]$ of $Q$ are isomorphic, hence an arbitrary isomorphism defines a *-operation $f: [a, b] \to [c, d]$.

A congruence relation $\Theta$ is called irreducible if it is a join-irreducible element of the congruence lattice. The smallest congruence relation $\Theta$ such that $a \equiv b (\Theta)$ will be denoted by $\Theta(a, b)$ and is called a principal congruence relation. If the congruence lattice is finite then every irreducible congruence relation is obviously principal.

THEOREM 1. Let $L$ be a finite distributive lattice. Then we can define on $Q$ *-operations $f_1, f_2, ...$ such that the congruence lattice of the partial algebra $Q_L = \langle Q; \lor, \land, f_i | i = 1, 2, ... \rangle$ will be isomorphic to $L$.

Proof. We prove the theorem by induction as follows: for each positive integer $n$ let $P(n)$ be the assertion that every distributive lattice of length $\leq n$ is isomorphic to $C(Q_L)$ for some partial algebra $Q_L$ defined on $Q$ with *-operations. If we take on $Q$ for each pair $0 \leq a_i < b_i \leq 1$ an arbitrary isomorphism $f_i: [a_i, b_i] \to [0, 1]$, then the corresponding partial algebra $Q_L = \langle Q; \lor, \land, f_i \rangle$ is obviously simple, i.e. $C(Q_L) \cong 2$. $P(1)$ is proved.

We shall show that $P(n-1)$ implies $P(n)$. Now let $L$ be any distributive lattice of length $n > 1$ and let $p$ denote a maximal irreducible element of $L$. Let $p_1, p_2, ..., p_k$ denote those irreducible elements of $L$ which are covered by $p$ in the poset of the irreducible elements. If $d$ denotes the join of all irreducible elements of $L$ different from $p$, then the length of the ideal $L_1 = \langle d \rangle$ is $n-1$. By the induction hypotheses
there exists a partial algebra \( Q_L \), defined on \( Q = [0, 1] \) with \(*\)-operations \( f_i \), such that \( C(Q_L) \cong L_1 \). Now we take the interval \([0, 2] \) of rational numbers and we define the partial algebra \( Q_L \) on the set \([0, 2] \).

The congruence relations of \( Q_L \), which correspond to \( p_i \in L_1 \) \((i = 1, \ldots, k) \) are irreducible, consequently principal, i.e. \( p_i \rightarrow \Theta(a_i, b_i) \) \((a_i < b_i)\). We distinguish two cases. First, if \( p \) is an atom \((k = 0)\), then we take for every pair \( 1 \leq a_i < b_i \leq 2 \) an arbitrary isomorphism \( f_i' : [a_i, b_i] \rightarrow [1, 2] \), and let

\[
Q_L = \langle [0, 2]; \lor, \land, f_1, f_i', f_i'^{-1} | i = 1, 2, \ldots \rangle.
\]

Then every congruence relation of \( Q_L \) is remade a congruence of \( Q_L \) if we take the rational numbers \( 1 < r \leq 2 \) as one element classes. For every \( 1 \leq a_i < b_i \leq 2 \) there is \( \Theta(a_i, b_i) = \Theta(1, 2) \) and therefore \( \Theta(1, 2) \) is an atom. Thus we have \( C(Q_L) \cong C(Q_{L_1}) \times \times 2 \cong L \). The intervals \([0, 1] \) and \([0, 2] \) are isomorphic, we can take also \( Q_L \) as a partial algebra defined on \([0, 1] \).

The second case is when \( k > 0 \). We take the following isomorphism:

\[
g_0 : [1, 2] \rightarrow \left[1 + \frac{k}{k+1}, \frac{2}{2}ight]
\]

is an arbitrary isomorphism with the property that \( \lim_{r \rightarrow \infty} g_0'(1) = 2 \). (For instance the mapping \( x \rightarrow (x + 2k)/(k + 1) \) is such an isomorphism);

\[
g_i : [a_i, b_i] \rightarrow \left[1 + \frac{i-1}{k+1}, 1 + \frac{i}{k+1}\right], \quad i = 1, 2, \ldots, k
\]

is an arbitrary isomorphism. The partial algebra \( Q_L \) is defined by

\[
Q_L = \langle [0, 2]; \lor, \land, f_1, g_0, g_i, g_i^{-1} | j = 1, 2, \ldots; i = 0, 1, \ldots, k \rangle.
\]

We shall prove that \( C(Q_L) \) is isomorphic to \( L \). To do this we prove some simple statements:

1. every \( \Theta \in C(Q_{L_1}) \) has an extension to a congruence relation \( \bar{\Theta} \) of \( Q_L \).

   Proof. Let \( \Theta \) be an arbitrary congruence relation of \( Q_{L_1} \), \( \Theta \) defines a reflexive and symmetric relation \( \Theta^* \) on \([0, 2] \):

   \[
u \equiv \nu(\Theta^*) \mathrm{iff} \begin{cases}
   \text{either } 0 \leq u, v \leq 1 \text{ and } u \equiv v(\Theta) \\
   \text{or } u = g_0^s g_i(x), \nu = g_0^s g_i(y), \quad g_i \leq x, y \leq b_i, \ x \equiv y(\Theta) \\
   \quad \text{for some integer } s \geq 0 \text{ and } 1 \leq i \leq k, \text{ where } g_0^0 \text{ is the identity map.}
\end{cases}
   \]

   Let \( \bar{\Theta} \) denote the transitive extensions of \( \Theta^* \). The restriction of \( \bar{\Theta} \) to \([0, 1] \) is \( \Theta \),
and $\vartheta$ is obviously a congruence relation of $Q_L$. $\vartheta$ is therefore an extension (the smallest extension) of $\vartheta$ to $Q_L$.

2. Every congruence relation $\vartheta(u, v)$, $0 \leq u \leq v < 2$ of $Q_L$ is the extension of a congruence relation $\vartheta \in C(Q_L)$. 

*Proof.* Let $x, y$ be two elements of the interval $[a_i, b_i]$ and let $\phi$ be an arbitrary congruence relation of $Q_L$. For two integers $s$ and $i$ ($1 \leq i \leq k$), $x = g_s g_i(x) \equiv y = g_s g_i(y)$ if and only if $x = g_i^{-1} g_0^{-s} (x') = g_i^{-1} g_0^{-s} (y') = y(\phi)$. $\vartheta(x, y) \in C(Q_L)$ is therefore an extension of $\vartheta(x, y) \in C(Q_L)$. If $1 \leq u < v < 2$, then there exists a natural number $m$ such that $u, v \leq g_0^m(1)$. There exists a finite chain

$$u = u_0 < u_1 < \cdots < u_t = v$$

such that for every $j (= 1, \ldots, t)$, $u_j \leq g_j^s(a_j, g_j^s(b_j))$ for some $s$ and $i$ ($1 \leq i \leq k$).

We have proved that $\vartheta(u_{j-1}, u_j)$ ($j = 1, \ldots, t$) is the extension of a congruence relation $\vartheta_j \in C(Q_L)$. Then $\vartheta(u, v)$ is obviously the extension of $\bigvee_{j=1}^t \vartheta_j$.

3. $\vartheta(u, 2) = \vartheta(1, 2)$ for every $1 \leq u < 2$, hence $\vartheta(1, 2)$ is irreducible.

*Proof.* Let $t$ be the last integer with $u \leq g_0^t(1)$. If $2 \equiv u(\phi)$ then $2 = g_0^{-t}(2) \equiv g_0^{-t}(u) = 1(\phi)$, hence $\vartheta(u, 2) \equiv \vartheta(1, 2)$. But $\vartheta(u, 2) \leq \vartheta(1, 2)$ is trivially satisfied and thus $\vartheta(u, 2) = \vartheta(1, 2)$. If $\vartheta(1, 2) = \vartheta_1 \vee \vartheta_2$ then there exists a sequence $2 = u_0 > u_2 > \cdots > u_t = 1$ such that for each $i$, $u_i \equiv u_{i+1}(\vartheta_i)$ or $u_i \equiv u_{i+1}(\vartheta_2)$. For instance $2 = u_2(\vartheta_1)$. But $\vartheta(1, 2) = \vartheta(u_2, 2)$ implies $\vartheta_1 = \vartheta(1, 2); \vartheta(1, 2)$ is therefore irreducible.

4. For an irreducible congruence relation $\vartheta \in C(Q_L)$ is $\vartheta(1, 2) \leq \overline{\vartheta}$ if and only if $\overline{\vartheta} \equiv \vartheta(a_i, b_i)$ for some $i \in \{1, \ldots, k\}$.

*Proof.* From $1 + (i-1)/(k+1) \equiv 1 + i/(k+1)$ ($\vartheta(1, 2)$) we get by the applications of $g_i^{-1} g_i^{-1}(1 + (i-1)/(k+1)) \equiv g_i^{-1}(1 + i/(k+1)) = b_i(\vartheta(1, 2))$ i.e.

$\overline{\vartheta(a_i, b_i)} \equiv \vartheta(1, 2)$. The statement "only if" is trivial.

1-4 imply that the poset of all irreducible congruence relations of $C(Q_L)$ is isomorphic to the poset of all irreducible elements of $L$, following $C(Q_L) \simeq L$.

3. Two preliminary constructions

**Lemma 1.** Let $N$ be a bounded distributive lattice. Then there exists a bounded modular lattice $M$ with the following properties:

(i) $M$ has three elements $a_1, a_2, a_3$ such that $0, a_1, a_2, a_3, 1$ form a sublattice isomorphic to $\mathcal{L}_3$ and $(a_1]$ is isomorphic to $N$;

(ii) for every congruence relation $\vartheta$ of $(a_i]$ ($i = 1, 2, 3$) there exists exactly one congruence relation $\overline{\vartheta}$ of $M$ such that for $\beta, \gamma \in (a_i]$ $\beta \equiv \gamma(\vartheta)$ iff $\beta \equiv \gamma(\overline{\vartheta})$. 

Proof. We take the set $M$ of all triples $(x, y, z)$ $(x, y, z \in N)$ with the property $x \land y = x \land z = y \land z$. $(x, y, z) \leq (x', y', z')$ means that $x \leq x'$, $y \leq y'$ and $z \leq z'$; then $M$ will be a poset. If $\beta = (x, y, z)$, $\gamma = (x', y', z') \in M$ then

$$(x \land x') \land (y \land y') = (x \land x') \land (z \land z') = (y \land y') \land (z \land z'),$$

therefore $\beta \land \gamma = (x \land x', y \land y', z \land z') \in M$. $M$ is therefore a $\land$-semilattice. It is easy to prove – using the distributivity of $N$ – that $\sup \{\beta, \gamma\} = \beta \lor \gamma$ exists and

$$\beta \lor \gamma = ((x \lor x') \lor ((y \lor y') \lor (z \lor z')),$$

$$\land ((y \lor y') \lor ((x \lor x') \lor (z \lor z')) \lor ((x \lor x') \lor (y \lor y'))).$$

The operations $\land$ and $\lor$ make $M$ into a lattice. This lattice was defined first in [5].

Now we prove the modularity of $M$. Let be $\gamma = (x, y, z)$, $\beta = (u, v, w)$ and $\alpha = (a, b, c)$, $\alpha > \beta$. The modularity means $z \land (\beta \lor \gamma) \leq \beta \lor (x \land \gamma)$. Take the first components of these elements: $a \land ((u \lor x) \lor ((v \lor y) \lor (w \lor z)))$ and $((u \lor (a \land x)) \lor (v \lor (b \land y)) \lor (w \lor (c \land z)))$. Then using the facts: $u \land w = u \land v$, $y \land z = x \land z$, $b \land v = v$, $c \land w = w$ and the distributivity of $N$ we can write:

$$a \land ((u \lor x) \lor ((v \lor y) \lor (w \lor z))) = [a \land (u \lor x)] \lor [a \land (v \lor y)] \lor (w \lor z)$$

$$= [u \lor (a \land x)] \lor (a \land v \lor w) \lor (a \land v \lor z)$$

$$\lor (a \land v \lor w) \lor (a \land v \lor z)$$

$$= u \lor (a \land x) \lor (a \land v \lor w) \lor (a \land v \lor z)$$

$$= u \lor (a \land x) \lor (a \land b \lor v \lor z) \lor (a \land y \lor w \lor c)$$

$$= u \lor (a \land x) \lor (a \land c \lor v \lor z) \lor (a \land y \lor w \lor b)$$

$$\leq u \lor (a \land x) \lor (a \land c \lor v \lor z) \lor (a \land y \lor w \lor b)$$

$$\lor (v \land w \lor (v \land c \lor z) \lor (v \land y \lor w) \lor (v \land y \lor c \lor z))$$

$$= (u \lor (a \land x)) \lor \{u \lor (b \land y) \lor (c \land z))\}.$$

The same inequality holds for the other components, hence $M$ is modular.

Let $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (0, 0, 1)$. Then $\alpha_i \land \alpha_j = (1, 1, 1)$, $\alpha_i \land \alpha_j = (0, 0, 0)$ if $i \neq j$ and so $\alpha_1, \alpha_2, \alpha_3, 1$ from $\mathcal{M}_5$. For the remaining statements of the lemma we refer to [5].

COROLLARY 1. The intervals $[0, \alpha_1]$ and $[0, \alpha_3]$ are projective; the corresponding unary algebraic function is $p_{1, 2}(x) = (x \lor \alpha_3) \land \alpha_2$, $[0, \alpha_1]$ and $[\alpha_1, 1]$ are projective; the corresponding function is:

$$q_1(x) = [(x \lor \alpha_3) \land \alpha_2] \lor \alpha_1 = p_{1, 2}(x) \lor \alpha_1.$$

Let us take two bounded lattices $L_1$ and $L_2$. Suppose that $L_1$ has a principal dual ideal $\mathcal{I}_1$, $L_2$ has a principal ideal $\mathcal{I}_2$ and $\mathcal{I}_1 \cong \mathcal{I}_2$. Let $\varphi: x \to x'$ denote this iso-
morphism. We can construct a lattice \( L \) as follows: \( L \) is the set of all \( x \in L_1 \) and \( x \in L_2 \), we identify \( x \) with \( x' \) for all \( x \in L_1; x \leq y \) have unchanged meaning if \( x, y \in L_1 \), or \( x, y \in L_2 \) and \( x \wedge y = x \vee y \) iff \( x \in L_1, y \in L_2 \) and there exists a \( z \in L_1 \) such that \( x < z \) in \( L_1 \), and \( z < y \) in \( L_2 \). Let \( \lor \), \( \land \), denote the join operations in \( L_1 \), and let \( 0_2 \) denote the zero of \( L_2 \). It is easy to see that in \( L \) \( \sup \{ x, y \} \) always exists and

\[
\sup \{ x, y \} = \begin{cases} x \lor y & \text{if } x, y \in L_1 \\ \left( x \lor 0_2 \right) \lor y & \text{if } x \in L_1 \text{ and } y \in L_2. \end{cases}
\]

By the duality we have that \( L \) is a lattice.

We denote \( L \) as follows: \( L = L_1 + L_2 (\phi \varphi_1 = \varphi_2) \).

**Lemma 2.** (Hall and Dilworth [3]). Let \( L_1 \) be a principal ideal and \( L_2 \) be a principal dual ideal of the lattice \( L \), such that \( L_1 \cup L_2 = L \) and \( L_1 \cap L_2 = \phi \). If \( L_1 \) and \( L_2 \) are modular lattices then \( L \) is a modular lattice too. If \( \Theta_1 \) and \( \Theta_2 \) are congruence relations of \( L_1 \) resp. \( L_2 \) such that \( \Theta_1 \upharpoonright [L_1 \cap L_2] = \Theta_2 \upharpoonright [L_1 \cap L_2] \), then there exists a congruence relation \( \Theta \) of \( L \) with the property \( \Theta \upharpoonright L_i = \Theta_i \) \( (i = 1, 2) \).

4. The construction

**Lemma 3.** Let \( K_0 \) be a modular lattice, which has the following properties:

1. \( K_0 \) has a dual ideal \( \mathcal{I}_0 \) isomorphic to \( Q \);
2. every congruence relation of \( K_0 \) is the smallest extension of a congruence relation of \( \mathcal{I}_0 \).

Let \( f: [a, b] \rightarrow [c, d] \), \( a, b, c, d \in \mathcal{I}_0 \) a *-operation on \( K_0 \), and let denote \( K_0^* = \langle K_0; \lor, \land, f \rangle \). Then there exists a realization \( K \) of \( K_0^* \) which is a modular lattice and has a dual ideal \( \mathcal{I} \) such that (1) and (2) are satisfied.

**Proof.** Let us take the following two lattices:

I. Let \( S \) denote the lattice \( M \) given in Lemma 1 if we set \( N = Q \). Then we take the intervals \( S_i = [0, x_i] \) and \( S_i^* = [x_i, 1] \) \( i = 1, 2 \) and the corresponding unary algebraic functions \( p_{i, i}(x) (i \neq j) \) and \( q_i \) given in the Corollary of Lemma 1.

II. \( T = Q \times Q \). Let be \( T_1 = [(0, 0), (1, 0)] \), \( T_2 = [(0, 0), (0, 1)] \) and \( T_1^* = [(1, 0), (1, 1)] \). Then it is obviously \( T_1 \cong T_2 \cong T_1^* \cong Q \). \( T_1 \) and \( T_2 \) are projective; the unary algebraic function is \( v_{1, 2}(x) = x \lor (0, 1) \).

We define \( A_0 = T + S (\psi T^1 = S_2) \) where \( \phi \) is an arbitrary isomorphism between \( T_1 \) and \( S_2 \). Let \( A \) be the lattice \( A_0 + T (\phi S^1 = S_2) \) where \( \psi \) is again an arbitrary isomorphism (Fig. 2). Let \( \sigma \) denote the zero of \( A \) and let \( e_1, e_2, e_3, e_4 \) be the elements given in Fig. 2 (\( e_1 \) is the element \( (1, 0) \), \( e_2 \) is the element \( (1, 0) \) of the second \( T \) component of \( A \); similarly \( e_3 \) is \( (1, 0) \) and \( e_4 \) is \( (0, 1) \) in the first \( T \) component of \( A \).
By Lemma 1 and 2 we get: for every congruence relation $\Theta$ of $(e_1]$ (resp. $[e_1]$) there exists exactly one congruence relation $\bar{\Theta}$ of $A$ such that for $x, y \in (e_1]$ (resp. $x, y \in [e_1]$) $x \equiv y(\bar{\Theta})$ iff $x \equiv y(\Theta)$. The intervals $[e_3, e_2]$ and $[e_4, e_3, v e_4]$ are projective, the corresponding unary function is $r_{12}q_1(x)$.

$(e_1]$ is the ordinal sum $S_1 + S_1 + S_1$ hence $(e_1]$ is isomorphic to $Q$. The same holds for the dual ideal $[e_4]$. Let now $\varrho$ be an arbitrary isomorphism $\varrho : S_0 \rightarrow (e_1]$ such that $\varrho(a) = e_3$ and $\varrho(b) = e_2$. Then we can define the lattice $B = K_0 + A (\varrho S_0 = (e_1])$. (See Fig. 3.)

The intervals $[a, b]$ and $[e_1, e_1 v e_2]$ are projective; we have namely the unary function $\lambda_1 = r_{12}q_1(\varrho(x))$.

Take $A$ in a second exemplar $A'$ ($x \rightarrow x'$ let be an isomorphism between $A$ and $A'$). If $\nu$ is an isomorphism $[e_4] \rightarrow (e_1']$ such that $\nu(\varrho(c) v e_4) = e_3'$ and $\nu(\varrho(d) v e_4) = e_2'$ then we can define $C = B + A' (\nu(e_4) = [e_1'])$. We have the lattice on Fig. 4.

Using the Lemma 2 we get: for every congruence relation $\Theta$ of $K_0$ there exists exactly one congruence relation $\bar{\Theta}$ of $C$ such that for $x, y \in K_0$ $x \equiv y(\bar{\Theta})$ iff $x \equiv y(\Theta)$. Similarly as in lattice $B$ we have: $[e_1 v e_4, e_1 v e_2]$ and $[c_v d]$ are projective; the corresponding function let $\lambda_2$.

Finally let us take the lattice $D = S + S (\mu S_2 = S_1)$ with an arbitrary isomorphism $\mu$ (Fig. 5) $[d_2, d_1]$ and $[d_3, d_2]$ are projective; let $\lambda_3$ denote the corresponding unary function.
The ideal \((d,]\) is isomorphic to \(Q\), hence it is isomorphic to the dual ideal \([e,]\) of 
\(C\). We shall define \(K = C + D\) with an isomorphism \(\phi: [e,] \rightarrow (d,]\) as follows: Let be 
\(\lambda^{-1} \lambda' \lambda^{-1} \lambda' f = f\) (Fig. 6). Then a congruence relation \(\Theta\) of \(K_0\) has an extension to 
\(K\) if \(\Theta\) is a congruence relation of \(K_0^*\) and every congruence relation of \(K\) is the extension 
of a congruence \(\Theta \in \Theta(K_0)\). \(K\) is a realization of \(K_0^*\). The dual ideal \(\mathcal{F}\) generated 
by \(e^* \in K\) satisfies obviously (1) and (2). This proves Lemma 3.

To prove our theorem let be \(K_0 = Q_\Lambda = \langle Q; \vee, \wedge, f_1, f_2, \ldots \rangle\). Applying the Lemma 3 
we get an extension \(K_1\) of \(K_0\) which realizes \(\langle Q; \vee, \wedge, f_1 \rangle\). We can extend \(K_1\) to \(K_2\) 
such that \(K_2\) realizes \(\langle Q; \vee, \wedge, f_1, f_2 \rangle\). From \(K_i\) we can get on similar way \(K_{i+1}\). By a 
direct limit procedure we get a lattice \(K\) which realizes \(Q_\Lambda\) and so \(C(K) \cong C(Q_\Lambda)\). This 
proves \(C(K) \cong L\).

Remark 1. A modular lattice satisfying the identity

\[ x \vee \bigwedge_{i=0}^{n} y_i = \bigwedge_{j=0}^{n} (x \vee \bigwedge_{i=0 \wedge j}^{n} y_i) \]

is called an \(n\)-distributive lattice. (See G. Bergman [1] and A. Huhn [4].) A. Huhn has proved that \(L\) is \(n\)-distributive iff \(L\) does not contain a sublattice \(B \cong 2^{n+1}\) and an
element $x$ such that $x$ is the relative complement of all atoms of $B$ in interval $[\inf B, \sup B]$. It is easy to show that the lattices $T$ and $S$ do not contain sublattice isomorphic to $2^3$, therefore we have:

**THEOREM 2.** Every finite distributive lattice is isomorphic to the congruence lattice of a 2-distributive lattice.

**Remark 2.** Let $K_0$ be an arbitrary bounded distributive lattice and $f_2$ ($x \in \Omega$) $*$-operations on $K_0$. In [6] it is proved\(^1\) that $K_0$ has a modular extension, which is a realization of $K_0^* = \langle K_0; \lor, \land, f_2 \rangle$.

**REFERENCES**


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\(^1\) G. Grätzer proved the same statement for an arbitrary distributive lattice $K_0$ with zero.