ON FINITELY GENERATED SIMPLE MODULAR LATTICES

by
E. T. SCHMIDT (Budapest)

0. Introduction

R. Wille asked the following question: does every finitely generated modular lattice contain a prime quotient? The answer is negative, as shown by the following

Theorem. There exists a finitely generated simple modular lattice of infinite length.

The proof is based on the method of [1].

1. Preliminaries

Let \( Q \) be the chain of all rational numbers \( \frac{k}{2^n}, 0 \leq k \leq 2^n, n = 0, 1, 2, \ldots \).

Lemma [1]. Let \( N \) be a bounded distributive lattice. Then there exists a bounded modular lattice \( M \) with the following properties:

(i) \( M \) has three elements \( u_1, u_2, u_3 \) such that 0, \( u_1, u_2, u_3, 1 \) form a sublattice isomorphic to \( \mathbb{Z}_3 \), (the five element modular but not distributive lattice) and \( (u_i) \) is isomorphic to \( N \);

(ii) for every congruence relation \( \Theta_i \) of \( (u_i) \) \( i = 1, 2, 3 \) there exists exactly one congruence relation \( \Theta \) of \( M \) such that for \( b, c \in (u_i) \) \( b = c(\Theta) \) iff \( b = c(\Theta_i) \).

We apply this lemma for the lattice \( Q = N \) and \( M \) will always denote this lattice getting from \( Q \).

The following lattice construction is due to Hall and Dilworth [2] (see also [1]). Let us take two bounded lattices \( L_1 \) and \( L_2 \). Suppose that \( L_1 \) has a principal dual ideal \( I_1 \), \( L_2 \) has a principal ideal \( I_2 \) and \( I_1 \cong I_2 \) by \( q : x \rightarrow x' \). We get a lattice \( L \) as follows: \( L \) is the set of all \( x \in L_1, y \in L_2 \), we identify \( x \) with \( x' \) for all \( x \in I_1, x \leq y \) have unchanged meaning if \( x, y \in L_1 \), or \( x, y \in L_2 \) and \( x < y, x, y \notin I_1 = I_2 \) iff \( x \in L_1, y \in L_2 \) and there exists a \( z \in I_1 \), such that \( x < z \) in \( L_1 \) and \( x < y \) in \( L_2 \). We denote \( L \) as follows:

\[ L = L_1 + L_2(qI_1 = I_2). \]
It is easy to see that for modular lattices \( L_1, L_2 \) the lattice \( L \) is modular, too ([1] Lemma 2).

2. The construction

In the lattice \( M \) the dual ideal \( \{u_i\} \) is isomorphic to \( \{u_j\} \) and both are isomorphic to \( Q \). Let us take \( M \) in five disjoint replicas \( M_1, M_2, M_3, M_4 \) and \( M_5 \). We denote the elements \( u_1, u_2, u_3, 0, 1 \) in \( M_i \) by \( u_1^i, u_2^i, u_3^i, 0^i, 1^i \). The \( \{u_i^i\} \subseteq M_1 \) is isomorphic to \( \{u_1^2\} \subseteq M_2 \), by the natural isomorphism. For a natural isomorphism we use always the symbol \( \varphi_n \).

We can define the following lattices

\[
A_0 = M_5 + M_4(\varphi_n[u_2^2] = (u_4^1)); \\
A_1 = M_1 + M_2(\varphi_n[u_1^2] = (u_2^2)); \\
A_2 = M_4 + A_1(\varphi_n[u_3^2] = (u_1^2));
\]

where

\[
\varphi^{-1}(x) = \begin{cases} 
\frac{x}{2} & \text{for } 0^1 \leq x \leq u_1^1, \\
\frac{x + 1}{2} & \text{for } 0^2 \leq x \leq u_2^1,
\end{cases}
\]

\[
K = A_0 + A_2(\varphi_n[1^1] = (0^2)).
\]

The poset of all \( 0^i, u_i^i, u_2^i, u_3^i, 1^i \in K \) \( (i = 1, 2, 3, 4, 5) \) is represented by Fig. 1.

In the lattice \( M \), \( u_i/0 \) and \( 1/0 \) are projective quotients, hence \( 1^3/0^3 \) and \( u_i^3/0^2 \) are projective in \( K \). We shall denote the corresponding algebraic function which maps \( 1^3/0^3 \) onto \( u_i^3/0^2 \) by \( f_0(x) \). Then

\[
f_0(x) = \left\{ \left[ \left[ \left( x \wedge u_2^1 \right) \vee u_3^2 \right] \wedge u_1^2 \right] \vee u_4^2 \right\} \wedge u_5^2 \
\]

Similarly, \( 1^3/0^3 \) and \( u_i^3/0^3 \) are projective, and there corresponds to the algebraic function \( g_0(x) \). \( f_0^{-1}(x) \) and \( g_0^{-1}(x) \) are inverse functions.

\[
u_i/0^3 \text{ is a sublattice of } 1^3/0^3, \text{ therefore the restrictions of } f_0, f_0^{-1}, g_0, g_0^{-1} \text{ define four unary partial operations } f_0, f_0^{-1}, g_0, g_0^{-1} \text{ on } 1^3/0^3. \text{ On the other hand, we have a natural isomorphism between } 1^3/0^3 \text{ and } Q. \text{ By this isomorphism we get the partial operations } f, f^{-1}, g, g^{-1} \text{ on } Q \text{ corresponding } f_0, f_0^{-1}, g_0, g_0^{-1}.
\]

By the definition of \( A_2 \) we have:

\[
f(x) = \frac{x + 1}{2}, \ x \in Q; \quad f^{-1}(x) = 2x - 1, \quad \frac{1}{2} \leq x \leq 1;
\]

\[
g(x) = \frac{x}{2}, \ x \in Q; \quad g^{-1}(x) = 2x, \quad 0 \leq x \leq \frac{1}{2}.
\]
We have to prove that $K$ is a finitely generated simple lattice. First we prove that $P = \{u_i, u'_i, u^i_i \mid i = 1, 2, \ldots, 5\}$ is a generating set. We can see that $f_0, f_0^{-1}, g_0, g_0^{-1}$ are defined by the elements of $P$, hence it is enough to prove that in the partial algebra

$$Q = \langle Q; \lor, \land, f, f^{-1}, g, g^{-1} \rangle$$

the subset $Q_0 = \{0, 1\}$ is a generating set. Let $Q_0$ be denote the subalgebra generated by $Q_0$, and let $Q_n$ be the following subset of $Q$:

$$Q_n = \left\{ \frac{0}{2^n}, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{k}{2^n}, \ldots, \frac{2^n}{2^n} \right\}.$$

The union $\bigcup_{n=0}^{\infty} Q_n$ is obviously $Q$. We prove by induction for $n$, that every $Q_n \subseteq Q_0$; $Q_0 \subseteq Q_n$ by the definition of $Q_0$. Now let $Q_n \subseteq Q_0$ and let $\frac{u}{2^{n+1}} \in Q_{n+1}$, where $0 \leq u \leq 2^{n+1}$. If $u \leq 2^n$ then $\frac{u}{2^n} \in Q_n$ and $\frac{u}{2^{n+1}} = g \left( \frac{u}{2^n} \right)$, hence $\frac{u}{2^{n+1}} \in Q_{n+1}$. In other case $2^n \leq u \leq 2^{n+1}$ we take $\frac{u - 2^n}{2^n} \in Q_n$ and apply the operation $f$ getting $\frac{u}{2^{n+1}} = f \left( \frac{u - 2^n}{2^n} \right) \in Q_{n+1}$.
By condition (ii) of the Lemma every congruence relation of \( K \) is the smallest extension of a congruence relation of the quotient \( 1^3/u_3^1 \). Therefore if \( Q \) is a simple partial algebra, \( K \) is a simple lattice. Let \( u, v, u < v \) be two elements of \( Q \) and let \( \Theta \) be a congruence relation such that \( u \equiv v(\Theta) \). Then there exist two integers \( k \) and \( n \) with the property

\[
u \leq \frac{k}{2^n} < \frac{k + 1}{2^n} \leq v.
\]

From \( u \equiv v(\Theta) \) we get \( \frac{k}{2^n} = \frac{k + 1}{2^n} (\Theta) \). If \( \frac{k + 1}{2^n} \leq \frac{1}{2} \) then we can apply \( g^{-1} \), hence

\[
\frac{k}{2^{n-1}} = g^{-1} \left( \frac{k}{2^n} \right) = g^{-1} \left( \frac{k + 1}{2^n} \right) = \frac{k + 1}{2^{n-1}} (\Theta).
\]

In the other case \( \frac{1}{2} \leq \frac{k}{2^n} \); from \( \frac{k}{2^n} = \frac{k + 1}{2^n} (\Theta) \) we get using \( f^{-1} \)

\[
\frac{k - 2^{n-1}}{2^{n-1}} = f^{-1} \left( \frac{k}{2^n} \right) = f^{-1} \left( \frac{k + 1}{2^n} \right) = \frac{k + 1 - 2^{n-1}}{2^{n-1}} (\Theta).
\]

By induction we have that from \( u \equiv v(\Theta) \) it follows \( 0 \equiv 1(\Theta) \), i.e., \( Q \) is a simple partial algebra.

This completes the proof of the theorem.

**Corollary.** There exist a finitely generated modular lattice which does not contain a prime quotient.

**Proof.** Let \( K \) be a finitely generated simple modular lattice of infinite length. If \( a/b \) is a prime quotient of \( K \) and \( c/d \) is an arbitrary quotient \( c \equiv d(\Theta(a, b)) \), hence there exists a finite chain \( d = z_0 < z_1 < \ldots < z_n = c \) such that \( z_i/z_{i-1} \) is weak projective into \( a/b \) (\( i = 1, 2, \ldots, n \)) and therefore \( z_i/z_{i-1} \) has a finite length. This implies that \( c/d \) has a finite length. This is a contradiction to the assumption that \( K \) is of infinite length.

**References**


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\frac{k - 2^{n-1}}{2^{n-1}} = f^{-1}\left(\frac{k}{2^{n}}\right) = f^{-1}\left(\frac{k + 1}{2^{n}}\right) = \frac{k + 1 - 2^{n-1}}{2^{n-1}} (\Theta).
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**References**


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