Standard sublattices

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0. Introduction

The concept of a standard ideal of a lattice was introduced in [1]. An ideal $S$ of a lattice $L$ is called standard if

$$I \land (S \lor K) = (I \land S) \lor (I \land K)$$

(1)

holds for any pair of ideals $I, K$ of $L$, where $\lor$ and $\land$ denote the lattice-operations of the ideal-lattice $I(L)$ of $L$.

This concept is a generalization of neutral ideals and has many useful properties. Standard ideals play the same role for lattices as invariant subgroups for groups. A congruence of a group is determined by any congruence-class. However, even this does not hold for congruences generated by standard ideals. So, we should take into consideration all "standard-like" possible congruence-classes.

The aim of this paper is to give a generalization of standard ideals for convex sublattices, called standard sublattices, and to prove that many important properties of standard ideals are also valid for standard sublattices.

1. The definition of a standard sublattice

We shall denote by $\cup$ and $\cap$ the set-theoretical and by $\lor$ and $\land$ the lattice-theoretical operations. $\emptyset$ denotes the empty set. The convex sublattice generated by a subset $A$ of the lattice $L$ will be denoted by $\langle A \rangle$. Let $A$ and $B$ be two (nonempty) subsets of the lattice $L$. Then we define

$$A \lor B = \langle \{a \lor b| a \in A, b \in B\} \rangle$$

$$A \land B = \langle \{a \land b| a \in A, b \in B\} \rangle,$$

i.e., $A \lor B$ and $A \land B$ are the convex sublattices of $L$ generated by the elements $a \lor b$ and $a \land b (a \in A, b \in B)$, respectively.

Let us remark, if $A$ and $B$ are both ideals (or both dual-ideals) then $A \lor B$ and $A \land B$ are exactly the join and the meet of $A$ and $B$ in the ideal-lattice. However, in the general case neither $A \leq A \lor B$ nor $A \land B \leq A$ are valid. For example, if $A = \{a\}$ and $B = \{b\}$ then both inequalities imply $A = B$.

Presented by G. Grätzer. Received March 6, 1974. Accepted for publication in final form December 16, 1974.
DEFINITION. A convex sublattice $S$ of a lattice $L$ is called a standard sublattice if
\begin{equation}
I \lor (S, K) = \langle I \lor S, I \lor K \rangle
\end{equation}
and
\begin{equation}
I \land (S, K) = \langle I \land S, I \land K \rangle
\end{equation}
hold for any pair $\{I, K\}$ of convex sublattices of $L$, whenever neither $S \lor K$ nor $I \land (S, K)$ are empty. (Thus, the word 'standard' implies convexity.)

PROPOSITION 1. For each $s \in L$, $\{s\} \lor K \neq \emptyset$ implies $s \in K$, yielding $\langle \{s\}, K \rangle = K$, $I \lor \{s\} \leq I \lor K$, and $I \lor \{s\} \leq I \lor K$. Thus, in (2) and (3) both the left and the right hand sides of the equations are $I \lor K$ and $I \lor K$, respectively.

Now, we are going to prove that we have indeed a generalization. To do this we need the following

LEMMA 1. For the convex sublattices $A$ and $B$ of the lattice $L$ the equalities
\begin{equation}
A \lor (B) = (A \lor B) ~ \text{and} ~ \langle A, (B) \rangle = (A \lor B)
\end{equation}
hold, where $(X)$ denotes the ideal generated by $X$.

Proof. $A \lor (B)$ implies, obviously, both $A \lor (B) \leq (A) \lor (B)$ and $\langle A, (B) \rangle \leq \langle (A), (B) \rangle = (A \lor B)$.

(i) $x \in (A \lor (B))$ implies $x \in (A)$ and $x \in (B)$, for $(A \lor (B)) = (A) \lor (B)$. Since the ideal $(A)$ consists of all elements of $L$ having an upper bound in $A$ we have $x \leq a$ for some $a \in A$. Hence $x = a \lor x \in A \lor (B)$.

(ii) Let $x \in (A)$, i.e., $x \leq a$ for some $a \in A$, and $y \in (B)$. Then we have, using the convexity of $\langle A, (B) \rangle$, by $y \leq x \lor y \leq a \lor y$ that $x \lor y \in \langle A, (B) \rangle$. Thus, $(A \lor (B))$ is the smallest convex sublattice containing all $x \lor y$ ($x \in (A)$, $y \in (B)$), is contained in $\langle A, (B) \rangle$.

PROPOSITION 2. An ideal $S$ of a lattice $L$ is standard if and only if it is a standard sublattice.

Proof. Let us assume, first, that the ideal $S$ is a standard sublattice of $L$. Then the ideals $I$ and $K$ are, of course, convex sublattices. Moreover $S \lor K \neq 0$ and $I \lor (S, K) \neq 0$ are, clearly, satisfied. Thus we have by (2) and (3)
\begin{equation}
I \lor (S, K) = \langle I \lor S, I \lor K \rangle ~ \text{and} ~ I \lor (S, K) = \langle I \lor S, I \lor K \rangle.
\end{equation}

$\langle (A, (B) \rangle = A \lor B$ for the ideals $A$ and $B$, i.e., we arrive at
\begin{equation}
I \lor (S \lor K) = (I \lor S) \lor (I \lor K) ~ \text{and} ~ I \lor (S \lor K) = (I \lor S) \lor (I \lor K).
\end{equation}
The first equality gives us, precisely, that \( S \) is a standard ideal. (The second equality is an obvious one.)

Let, conversely, \( S \) be a standard ideal.

Using the obvious equality \((I \land K) = (I \land (K))\) valid for any subset \( I \) and \( K \) of \( L \), we have for the convex sublattice \( I \) and \( K \) of \( L \), by Lemma 1:

\[
I \land (S \lor (K)) = (I \land S) \land (I \land (K))
\]

and

\[
(I \land S, I \land K) = (I \land S, I \land K) = (I \land S) \lor (I \land K) = ((I \land S) \lor (I \land K)).
\]

The standard ideal property for \( S \) yields (2).

We claim, now, that (3) is valid for every ideal \( S \) of \( L \).

\( S \subseteq (S, K) \) and \( K \subseteq (S, K) \) imply \((I \lor S, I \lor K) \subseteq (I \lor S, K) \). By Lemma 1 \((S, K) = S \lor (K)\), i.e., \( I \lor (S, K) \) is, clearly, the convex sublattice generated by the elements of the form \( x \lor (s \land y) \) where \( x \in I, s \in S, y \in t \) for some \( t \in K \). By convexity

\[
x \lor (s \land y) \land (x \lor t) \in (I \lor S, I \lor K)
\]

and

\[
x \lor (s \land y) \lor (x \lor t) \subseteq x \lor (s \land y) \lor (x \lor t)
\]

imply \( x \lor (s \land y) \in (I \lor S, I \lor K) \).

Using the convexity of \((I \lor S, I \lor K) \) again we have \( I \lor (S, K) \subseteq (I \lor S, I \lor K) \) which finishes the proof.

Now, we prove that standard sublattices have similar characterizations to those of standard ideals in [1].

**THEOREM 1.** The following four conditions are equivalent for each convex sublattice \( S \) of a lattice \( L \).

(a) \( S \) is a standard sublattice.

(b) Let \( K \subseteq L \) be any convex sublattice of \( L \) such that \( K \land S \neq \emptyset \). Then, to each \( x \in (S, K) \) there exist \( s_1, s_2 \in S, a_1, a_2 \in K \) such that:

\[
x = (x \land s_1) \lor (x \land a_1) = (x \lor s_2) \land (x \lor a_2).
\]

(b') Let \( K \) be as before. Then, for each \( S \) and to each elements \( x \in (S, K) \) and to each \( s_2, s_1 \in S \) there are elements \( s_1, s_2 \in S, a_1, a_2 \in K \) such that:

\[
x = (x \land s_1) \lor (x \land (a_1 \lor s_2)) = (x \lor (s_2 \land a_2) \land (x \lor (a_2 \land s_1))).
\]

(c) The relation \( \theta [S] \) on \( L \) defined by:

\[
\theta [S] (x, y) \iff x \lor (s \land y) \in (I \lor S, I \lor K)
\]
\[ x \equiv y(\theta[S]) \text{ if and only if} \]
\[ x \land y = ((x \land y) \lor t) \land (x \lor y) \quad \text{and} \quad x \lor y = ((x \lor y) \land s) \lor (x \land y) \]

with suitable \( t, s \) in \( S' \) is a congruence relation.

**Proof.** We will prove the equivalence of the four conditions cyclically.

(a) implies (β). Let \( K \cap S \neq \emptyset \) and let \( x \in \langle S, K \rangle \). For \( I = \{x\} \) the relation \( I \cap \langle S, K \rangle \) is satisfied. \( x \) is, clearly, the greatest element of \( I \cap \langle S, K \rangle \), i.e., by (2) the greatest element of \( \langle I \land S, I \land K \rangle \). Thus, \( x = (x \land s_1) \lor (x \land a_1) \) with suitable \( s_1 \in S, a_1 \in K \).

The dual property follows, similarly, from (3).

(β) implies (β'). Let \( K' = \langle s \rangle, K \rangle \) where \( s \) is an arbitrary element of \( S \). Then \( s \in K' \) implies \( S \cap K' \neq \emptyset \). Further, \( K \leq K' \leq \langle S, K \rangle \) yields \( \langle S, K \rangle = \langle S, K' \rangle \). Thus, for any \( x \in \langle S, K \rangle \) the equality
\[ x = (x \land s_1) \lor (x \land a_1) \quad (s_1 \in S, a_1 \in K') \]
holds, by (β). However \( a_1 \leq a_1 \lor s (a_1 \in K) \) implies \( x \leq (x \land s_1) \lor (x \land (a_1 \lor s)) \leq x \lor x = x \), i.e.,
\[ x = (x \land s_1) \lor (x \land (a_1 \lor s)). \]

(4)

We have dually, for an arbitrary \( s' \in S \) the equality
\[ x = (x \lor s_2) \land (x \lor (a_2 \lor s')) \quad (s_2 \in S, a_2 \in K). \]

(5)

**Remark.** Substituting in (β') \( s_1, s_2', a_1, a_2 \) by \( s_3, s_4', a_3, a_4 \), respectively, where \( s_3 \geq s_1, s_4' \leq s_2', a_3 \geq a_1, a_4 \leq a_2 (s_3, s_4' \in S, a_3, a_4 \in K) \) we also get equality.

**Proof.** It is enough, by duality, to deal only with \( s_3 \) and \( a_3 \). Monotony implies:
\[ x = (x \land s_1) \lor (x \land (a_1 \lor s_2)) \leq (x \land s_3) \lor (x \land (a_3 \lor s_2)) \leq x \lor x = x, \]
proving the statement.

(β') implies (γ). Let \( \theta[S] \) be defined as follows: \( x \equiv y(\theta[S]) \), for \( x \geq y \), if and only if \( y = (y \lor t) \land x \) and \( x = (x \land s) \lor y \) hold for suitable \( s, t \) in \( S' \).

Let us mention, that we may choose \( s \) and \( t \) such that \( s \geq t \) holds, because of the monotony. It is not too hard to verify that \( \theta[S] \) is an equivalence relation (see [2] p. 24.). We shall prove, that \( x \land z \equiv y \land z(\theta[S]) \) and \( x \lor z \equiv y \lor z(\theta[S]) \) are also valid for every \( z \in L \). Since the definition of \( \theta[S] \) is self-dual it is enough to prove only the first statement.

\( y \leq x \) implies \( y \land z \leq x \land z \) thus the trivial inequality \( y \land z \leq t \lor (y \land z) \) gives us \( y \land z \leq (t \lor (y \land z)) \land (x \land z) \). On the other hand
\[ (t \lor (y \land z)) \land (x \land z) \leq (t \lor y) \land x \land z = (t \lor (y \land x)) \land z = y \land z, \]
i.e.,
\[ y \land z = ((y \land z) \lor t) \land (x \land z). \]

Now, let \( K \) be the convex sublattice \( \langle t \land y \land z, y \rangle \). We have \( s \land t \land y \land z \in \langle S, K \rangle \), for \( s \in S \), \( t \land y \land z \in K \). By the convexity of \( \langle S, K \rangle \) the inequalities

\[ t \land y \land z \leq t \land y \leq t \land x \leq s \land x \leq s \]

imply \( s \land x \in \langle S, K \rangle \) yielding \( x = (s \land x) \lor y \in \langle S, K \rangle \). Thus, the convexity of \( \langle S, K \rangle \) and the inequalities

\[ t \land y \land z \leq y \land z \leq x \land z \leq x \]

imply \( y \land z, x \land z \in \langle S, K \rangle \).

Since \( t \in S \), we have, by \((\beta')\), elements \( s' \in S \), \( a_t \in K \) such that

\[ x \land z = ((x \land z) \land s') \lor (x \land z \land (a_t \lor t)). \]

As \( y \) is the greatest element of \( K \), we obtain, by the remark, that

\[ x \land z = ((x \land z) \land s') \lor ((x \land z) \land (y \lor t)) = ((x \land z) \land s') \lor ((y \lor t) \land x) \land z = (x \land z) \land s' \lor (y \land z). \]

Hence, \( \theta[S] \) is a congruence relation.

\((\gamma)\) implies \((a)\). It is enough to prove \((2)\). \( S, K \leq \langle S, K \rangle \) implies \( \langle I \land S, I \land K \rangle \leq \langle I \land S, I \land K \rangle \), i.e. we have to prove \( I \land \langle S, K \rangle \leq \langle I \land S, I \land K \rangle \). First we prove that each \( u \in I \cap \langle S, K \rangle \) is contained in \( \langle I \land S, I \land K \rangle \). Let \( v \) be an element of the nonempty set \( S \cap K \). \( \langle S, K \rangle \) is, obviously, the set of all elements \( y \) with \( s_1 \land k_1 \leq y \leq s_2 \lor k_2 \) \((s_1, s_2 \in S, k_1, k_2 \in K)\). Moreover, monotonicity implies that we may also suppose \( s_1 \leq v \leq s_2 \) and \( k_1 \leq u \leq k_2 \). The same hold for \( u \), for it belongs to \( \langle S, K \rangle \). \( s_t \equiv s_t \theta[S] \) implies \( k_2 \lor s_2 \equiv k_2 \lor s_2 \theta[S] \). Then, there exists by \((\gamma)\) an \( s \in S \) such that

\[ u = (u \land s) \lor (u \land k_2). \]

\[ u \in I \Rightarrow u = (u \land s) \lor (u \land k_2) \in \langle I \land S, I \land K \rangle. \]

Since \( I \land \langle S, K \rangle \) is the smallest convex sublattice containing all elements of the form \( i \land y \) \((i \in I, y \in \langle S, K \rangle)\), it is enough to prove that all of these elements are in \( \langle I \land S, I \land K \rangle \). Since \( I \cap \langle S, K \rangle \neq \emptyset \) it contains an element \( u \). We may choose \( s_1 \in S, k_1 \in K \) such that \( s_1 \land k_1 \leq y \) as we have seen. Thus:

\[ i \land (s_1 \land k_1) \leq i \land y \leq (i \lor u) \land (y \lor u). \]

It is enough to prove, by convexity, that

\[ i \land (s_1 \land k_1) \in \langle I \land S, I \land K \rangle \quad \text{and} \quad i \land y \in \langle I \land S, I \land K \rangle \quad \text{for} \quad u \leq i, u \leq y, \]
since 
\[ u \in I \cap \langle S, K \rangle \quad \text{implies} \quad i \lor u \in I, y \lor u \in \langle S, K \rangle. \]

\[ i \land (s_1 \land k_1) = (i \land s_1) \land (i \land k_1) \] proves that this element belongs to \( \langle I \land S, I \land K \rangle \).

\[ u \leq i \lor y \leq i \quad \text{and} \quad u \leq i \land y \leq y \] imply that \( i \land y \in I \cap \langle S, K \rangle \), i.e., \( i \land y \) is an element of \( \langle I \land S, I \land K \rangle \). Hence, the theorem is proven.

**COROLLARY 1.** If \( S \) is a standard sublattice then \( S \) is a congruence class by the congruence relation \( \theta[S] \).

**Proof.** Let \( x \equiv y(\theta[S]), x > y \). We have to prove that if one of these elements belongs to \( S \) then both of them are in \( S \). By the self-dual definition of standard sublattice we may assume \( y \in S \). By condition (\( \gamma \)) \( y = (y \lor t) \land x \) and \( x = (x \land s) \lor y \) with suitable \( s, t \in S \). Then \( x = (x \land s) \lor y \leq (x \land (y \lor s)) \lor y \leq x \), i.e., \( x = (x \land (y \lor s)) \lor y \). Trivially \( y \leq x \land (y \lor s) \leq y \lor x \) and \( y, y \lor s \in S \). Hence, by the convexity of \( S, x \land (y \lor s) \in S \) yielding \( x = (x \land (y \lor s)) \lor y \in S \).

Let \( S \) be a standard sublattice of the lattice \( L \). Then \( L/S \) denotes the lattice \( L/\theta[S] \).

**COROLLARY 2.** Let \( S \) and \( T \) be two standard sublattices. Then \( S \cap T \) is either a standard sublattice or it is empty.

**Proof.** We may assume that \( S \cap T \) contains an element \( u \). Let us suppose that \( x \equiv y(\theta[S]) \) and \( x \equiv y(\theta[T]) \) where \( x > y \). Then we have \( x = (s_1 \land x) \lor y \) with a suitable \( s_1 \in S \) which may be supposed to be greater then \( u \). On the other hand \( x \equiv y(\theta[T]) \) implies \( s_1 \land x \equiv s_1 \land y \) by the monotony. Hence, (\( \gamma \)) implies \( s_1 \land x = (t_1 \land (s_1 \land x)) \lor (s_1 \land y) \) with a \( t_1 \geq u \) in \( T \). Consequently \( x = (s_1 \land x) \lor y = ((t_1 \land s_1) \land x) \lor y \). But \( t_1, s_1 \geq u \) yield \( t_1 \land s_1 \geq u \), i.e., \( t_1 \land s_1 \in T \cap S \). The duality finishes the proof.

**COROLLARY 3.** The meet of a standard ideal and a standard dual ideal is a standard sublattice.

**Proof.** By Proposition 2 and by the duality all standard ideals and standard dual ideals are standard sublattices. Corollary 2 completes the proof.

**Remark.** The converse of Corollary 3 is not true. For example in \( N_4 \) there are one-element subsets which are not the meet of a standard ideal and of a standard dual ideal. Proposition 1 proves our statement. We can prove more. We define on the set \( \{a_0, ..., a_n; b_0, ..., b_m; c_0, ..., c_p\} \) the following partial order:

\[ a_0 < b_0, \quad a_{i+1} < a_i, \quad b_i < b_{i+1}, \quad a_{i+1} < c_i < b_{i+1}. \]

It is easy to see that we have a subdirectly irreducible lattice, where \( \theta(a_0, b_0) \) is the smallest congruence. This lattice has neither 0 nor 1, i.e., for each standard ideal or
standard dual ideal the congruence class containing $a_0$ must also contain $b_0$. Thus the standard sublattice \{$a_0\}$ is not even the meet of two congruence-classes generated by a standard ideal and by a dual standard ideal.

2. Properties of standard sublattices

Firstly, we prove two Lemmas.

**LEMMA 2.** Let $S$ be a standard sublattice and $I$ be an arbitrary convex sublattice of the lattice $L$ such that $I \cap S \neq \emptyset$. Then $I \cap S$ is a standard sublattice of the lattice $I$.

**Proof.** $I \cap S$ is obviously a convex sublattice of $I$. To prove that $I \cap S$ is standard we use condition (β). Each convex sublattice $K$ of $I$ is, clearly, a convex sublattice of $L$ itself. Thus, by (β) each $x \in \langle S \cap I, K \rangle$ is to be written in the form

$$x = (x \wedge s) \vee (x \wedge a) \quad (s \in S, a \in K),$$

since $K \cap S = (K \cap I) \cap S = K \cap (I \cap S)$ is not empty.

We may assume, by monotony, that both $s \geq u$ and $a \geq u$, where $u$ is a given element of $K \cap (S \cap I)$. Then we have for $s' = (x \vee u) \wedge s$:

$$u = (x \vee u) \wedge u \leq s'; \quad s' \leq x \vee u; \quad s' \leq s.$$  

$u \in S \cap I, x \vee u \in I, s' \in S$ imply $s' \in S \cap I$. Hence, by $x \wedge s' = x \wedge (x \vee u) \wedge s = x \wedge s$,

$$x = (x \wedge s') \vee (x \wedge a) \quad (s' \in S \cap I, a \in K)$$

yields (β) in $I$. The duality finishes the proof.

**LEMMA 3.** Let $x \rightarrow x'$ be a homomorphism of $L$ onto $L'$ and let $S$ be a standard sublattice of $L$. The homomorphic image $S'$ of $S$ is a standard sublattice of $L'$.

**Proof.** We shall prove (β) for $S'$. The coimage $K$ of an arbitrary convex sublattice $K'$ of $L'$ is, obviously, a convex sublattice of $L$. $K' \cap S' \neq \emptyset$ implies $K \cap S \neq \emptyset$. Each $y' \in \langle S', K' \rangle$ has, clearly, a coimage $x \in \langle S, K \rangle$ for which, by (β)

$$x = (x \wedge s) \vee (x \wedge a) \quad (s \in S, a \in K)$$

holds. Then, $x' = y', s' \in S', a' \in K'$ proves the first statement of (β) for $S'$. The proof is completed by duality.

**THEOREM 2.** (The first isomorphism theorem). Let $L$ be a lattice, $S$ a standard
sublattice and I a convex sublattice of L such that $S \cap I \neq \emptyset$. Then $S \cap I$ is a standard sublattice of I and:

$$\langle I, S \rangle / S \cong I / (I \cap S).$$

**Proof.** The first statement was proved in Lemma 2. Using the first isomorphism theorem for universal algebras it remains to prove that every congruence class of $\langle I, S \rangle$ may be represented by an element of I. Indeed, if $x \in \langle I, S \rangle$ then, by (β), $x = (x \land s_1) \lor (x \land a_1) = (x \lor s_2) \land (x \lor a_2)$ ($s_1, s_2 \in S, a_1, a_2 \in I$) and choosing any $u$ in $S \cap I$ we may suppose that $s_2 \leq u \leq s_1, a_2 \leq u \leq a_1$. Then, 

$$x = (x \land s_2) \lor (x \land a_1) \equiv (x \land s_2) \lor (x \land a_1) = x \land a_1(\theta[S]), \text{ for } s_2 \leq a_1,$$

and, similarly, $x = x \lor a_2(\theta[S])$. For $y = (x \land a_1) \lor a_2$ we have $a_2 \leq y \leq a_1$ yielding $y \in I$ and $x \equiv x \lor a_2 \equiv (x \land a_1) \lor a_2 = y(\theta[S])$ proving the theorem.

**THEOREM 3** (Second isomorphism theorem). Let $L$ be a lattice $S$ a convex sublattice and $T$ a standard sublattice of $L$ such that $T \leq S$. Then $S$ is standard in $L$ if and only if $S / T$ is standard in $L / T$ and in this case the isomorphism $L / S \cong (L / T) / (S / T)$ holds.

**Proof.** If $S$ is standard then $S / T$ is standard in $L / T$ by Lemma 3. The converse is proved in the same way as it is in [1] for standard ideals. The second isomorphism theorem for universal algebras finishes the proof.

It has been proved (see [1]) that $L$ is a distributive lattice whenever every ideal of it is standard. A similar statement holds for standard sublattices.

**THEOREM 4.** Let $u$ be an element of the lattice $L$. If every convex sublattice containing $u$ is standard then $L$ is a distributive lattice.

**Proof.** We shall prove that distributivity is implied whenever the ideals and the dual ideals containing $u$ are standard. Let, namely, $L_1 = L/[u]$ and $L_2 = L/[u]$. The condition and Lemma 3 imply that each ideal of $L_1$ and each dual ideal of $L_2$ is standard. Thus, both $L_1$ and $L_2$ are distributive. If $a \leq b$ have the same image both in $L_1$ and in $L_2$ then exist $p \leq u \leq q$ such that $b = a \lor p$ and $a = b \land q$, since both $[u]$ and $[u]$ are standard. Thus,

$$a = b \land q = b \land ((b \land q) \lor q) = b \land (a \lor q) \geq b \land (a \lor p) = b \land b = b,$$

proving that $L$ is a subdirect product of the two distributive lattices $L_1$ and $L_2$. Hence, $L$ is itself distributive.

**THEOREM 5.** In a relatively complemented lattice every congruence class is a standard sublattice.
Proof. Let $L$ be a relatively complemented lattice and let $\theta$ be a congruence relation on $L$. Let, further, $S$ denote the congruence class containing a given element $a$ of $L$.

For $x \leq y$, $x \equiv y(\theta)$ let $x'$ denote the relative complement of $x$ in the interval $[x \land a, y]$ and let $y'$ denote the relative complement of $y$ in the interval $[x, a \lor y]$. From $x \equiv y(\theta)$ follows $x \land a = x \land x' \equiv y \land x' = y'(\theta)$ yielding $a = a \lor (x \land a) \equiv a \lor x'(\theta)$. Hence, $t = a \lor x'$ is an element of $S$ and so is, dually, the element $s = a \land y'$. Further, $x' \leq y \land (a \lor x') = y \land t$ implies $y = x' \lor x \leq (y \land t) \lor x \leq y$ proving $y = (y \land t) \lor x$. We get $x = (x \lor s) \land y$ dually. Thus, condition (γ) of Theorem 1 is satisfied for $S$, i.e., $S$ is standard.

Each standard sublattice is a class of a congruence relation. If the lattice is relatively-complemented this relation is unique and all classes are standard sublattices, i.e., $\theta[S] = \theta$ holds for each congruence class $S$ of $\theta$.

In the following example we will give a congruence $\theta$ of a lattice $L$ such that $\theta[S] = \theta$ holds for each congruence class $S$ of $\theta$ though none of these classes are standard.

Let $N^{(n)}$ denote a family of lattices isomorphic to $N_5$ for each integer $n$. The elements of $N^{(n)}$ will be denoted by $o_n$, $a_n$, $b_n$, $c_n$, $i_n$, respectively, where $o$ is the smallest element $i$ is the greatest element and $a < b$. There is an amalgam $L$ of these lattices such that $o_n = c_{n-1}$ and $a_n = i_{n-1}$ and $L$ contains no further elements. Now, the classes $S_n = \{o_n, a_n, b_n\}$ are classes of a congruence relation having the desired property.

REFERENCES


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