LATTICES GENERATED BY PARTIAL LATTICES

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Dedicated to I. Rédei on his 75-th birthday

1. INTRODUCTION

Let $M$ be a lattice and $D$ be a distributive lattice with 0 and 1. If $a/b$ is a prime quotient of $M$ then we can define a partial lattice $M(aDb)$ as follows: $M(aDb)$ is the set-theoretical union of $M$ and $D$, and we identify $a$ with 1 and $b$ with 0; $M$ and $D$ are sublattices and $m \lor d$, $m \land d$ ($m \in M$, $d \in D$) are defined only for $d \in M \cap D = \{a, b\}$. Shortly speaking we put the distributive lattice $D$ into the prime quotient $a/b$ of $M$. We will show that in every equational class $\mathcal{X}$ containing $M$ there exists a lattice which contains a relative sublattice isomorphic to $M(aDb)$. In this paper we define a special lattice $M[D]$ (an extension of $M$ by $D$) containing the relative sublattice $M(aDb)$. The construction is a generalization of that given by R.W. Quackenbush [7] for the case if $M$ is a bounded distributive lattice. Let $M_3$ be the five-element modular, non-distributive lattice, $M_3[D]$ was defined in [9] and later discussed by A. Mitschke and R. Wille [5]. Let $\tilde{M}(aDb)$ be the sublattice of $M[D]$ generated by $M(aDb)$. We prove that the quotient sublattice of $a/b$ in $\tilde{M}(aDb)$ (and in $M[D]$) is isomorphic to $D$, and $\Theta(a/b)/\omega$ in $\tilde{M}(aDb)$
is isomorphic to \( \Theta(D) \). For modular \( M \), these results have been proved independently by R. Freese [3]. Finally we give a shorter proof of the main theorem of [5].

2. THE FINITE CASE

Let first \( D \) be a finite distributive lattice with the ultrafilters \( Q_1, Q_2, \ldots, Q_n \). (By the definition of the ultrafilter \( Q_i \neq D \)).

For an arbitrary lattice \( M \) we define a special subdirect power \( M[D] \) of \( M \) as follows: \( M[D] \) contains all \( d = (d_1, d_2, \ldots, d_n) \in M^n \) for which \( Q_i \supseteq Q_j \) implies \( d_i \supseteq d_j \). \( M[D] \) is obviously a sublattice of \( M^n \), hence a subdirect power of \( M \). Let \( X \) be the poset of all ultrafilters of \( D \). Then \( M[D] \) is the lattice of all monotone maps of \( X \) into the lattice \( M \).

Before we discuss some interesting properties of this lattice, let us take too examplars for \( M[D] \). Let \( D \) be the three-element chain \( 3 \). We shall consider \( M = M_3 \) (with the elements \( 0, a_1, a_2, a_3, 1 \)), and in the second case \( M = N_5 \), the five-element non-modular lattice (with the elements \( 0, 1, a, b, c; a > b \)). The corresponding lattices \( M_3[3] \) and \( N_5[3] \) are:

\[
\begin{align*}
M_3[3] & \quad (1, 1) \\
& \quad (1, a) \\
& \quad (a, a) \\
& \quad (a, b) \\
& \quad (a, 0) \\
& \quad (b, b) \\
& \quad (0, 0)
\end{align*}
\]

\[
\begin{align*}
N_5[3] & \quad (1, 1) \\
& \quad (1, a) \\
& \quad (a, a) \\
& \quad (a, b) \\
& \quad (a, 0) \\
& \quad (b, b) \\
& \quad (0, 0)
\end{align*}
\]
If \( 2 \) is the two-element distributive lattice then \( M[2] \cong M \). The correspondence
\[
m \mapsto f_m = (m, m, \ldots, m) \quad (m \in M)
\]
is the canonical embedding of \( M \) into \( M^n \) and by the definition of \( M[D] \) every \( f_m \) belongs to \( M[D] \). Let \( a/b \) a prime quotient of \( M \). We shall show that the corresponding quotient \( f_a/f_b \) of \( M[D] \) is isomorphic to \( D \). But by the Birkhoff – Stone representation theorem we have for every \( d \in D \) the correspondence \( d \mapsto g_d = (d_1, d_2, \ldots, d_n) \in 2^n \), where \( d_i = 1 \) if \( d \in Q_i \) and \( d_i = 0 \) if \( d \notin Q_i \). If \( d_i \neq d_j \) then \( d_i = 0, d_j = 1 \) hence \( d \not\in Q_j \), \( d \in Q_i \) and therefore \( Q_i \nsubseteq Q_j \); hence \( Q_i \nsubseteq Q_j \) implies \( d_i \nleq d_j \). We define \( g_d(a/b) = (y_1, y_2, \ldots, y_n) \in f_a/f_b \), where \( y_i = a \) if \( d_i = 1 \) and \( y_i = b \) otherwise. Then \( d \mapsto g_d(a/b) \in M[D] \) is an isomorphism between \( D \) and \( f_a/f_b \). The elements \( f_m \ (m \in M) \) and \( y \ (y \in f_a/f_b) \) form a relative sublattice isomorphic to \( M(aDb) \), hence

**Proposition 1.** \( M(aDb) \) is isomorphic to a relative sublattice of \( M[D] \).

Let \( \mathcal{X} \) be an equational class containing \( M \) (we assume that \( |M| > 1 \)).

**Proposition 2.** The free lattice \( F_{\mathcal{X}}(M(aDb)) \) over \( \mathcal{X} \) generated by \( M(aDb) \) exists.

Let \( \hat{M}(aDb) \) be the sublattice of \( M[D] \) generated by \( M(aDb) \). \( M[D] \) is a subdirect power of \( M \) hence

**Proposition 3.** Every congruence relation of \( a/b (= f_a/f_b) \) can be extended to \( \hat{M}(f_a/f_b) (= \hat{M}(aDb)) \), hence in \( \hat{M}(aDb) \) \( \Theta(a, b)/\omega \) is isomorphic to \( \Theta(D) \).

Let \( M \) be a bounded lattice. Then the \( g_d \) defined above can be taken as an element of \( M[D] \).

**Proposition 4.** If \( M \) is a bounded lattice then \( M[D] \) is generated by
\[
\{ f_m \mid m \in M \} \cup \{ g_d \mid d \in D \}.
\]
Proof. (See [4]). Let $h = (h_1, h_2, \ldots, h_n) \in M[D]$. Let $X_{h_i} = \{Q_j \mid h_j \geq h_i\}$. Then $X_{h_i}$ is an increasing subset of $X$ (a subset $E$ of $X$ is increasing if $x \in E$, $y \geq x$ imply $y \in E$). Hence there exists a unique element $e_i \in D$ such that $X_{h_i} = \{Q_j \mid e_i \in Q_j\}$ ($e_i$ is the minimal element of the intersection $\cap Q_j$ of all $Q_j \in X_{h_i}$). We prove that $h = \bigvee_{i=1}^n (f_{h_i} \land g_{e_i})$. Let us take:

$$\bigvee_{i=1}^n (f_{h_i} \land g_{e_i})(Q_j) = \bigvee \{h_i \mid e_i \in Q_j\} =$$

$$= \bigvee \{h_i \mid Q_j \in X_{e_i}\} = \bigvee \{h_i \mid h_j \geq h_i\} = h_j.$$

3. THE GENERAL DEFINITION OF $M[D]$

$D$ will denote the equational class of bounded distributive lattices. If $D \in D$, then the set $X$ of all ultrafilters of $D$ becomes a compact totally order disconnected space by identifying $X$ with the set $\text{Hom}_D(D, 2)$ of homomorphism onto 2. Let $M$ be an arbitrary lattice.

Definition. $\mathcal{C}(X, M)$ is the lattice of all continuous monotone maps of the compact totally order disconnected space $X$ into the discrete space $M$.

If $D$ is a finite distributive lattice then by the definition of $M[D]$ in the previous paragraph we obtain that $M[D] = \mathcal{C}(X, M)$. Therefore we define in the general case: $M[D]$ is the lattice $\mathcal{C}(X, M)$.

Remark. We give a motivation for the definition of $M[D]$. Let $M$ be a bounded distributive lattice. In [7] R.W. Quackenbush has defined $M[D]$ as follows: $M[D]$ is the subalgebra of $M^X$ generated by $\{f_m \mid m \in M\} \cup \{g_d \mid d \in D\}$, where for all $Q \in X$ $f_m(Q) = m$ and

$$g_d(Q) = \begin{cases} 1 & \text{if } d \in O, \\ 0 & \text{if } d \not\in Q. \end{cases}$$

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Brian A. Davey [2] has shown, that (for distributive $M$) $M[D]$ is isomorphic to $\mathcal{C}_< (X, M)$. (The proof is essentially the same as the proof of Proposition 4). The proof does not use the distributivity of $M$, we have therefore

**Proposition 5.** If $M$ is a bounded lattice, then $M[D]$ is the sublattice of $M^X$ generated by

$$\{f_m \mid m \in M\} \cup \{g_d \mid d \in D\}.$$  

The set $\{f_m \mid m \in M\}$ is a sublattice of $M[D]$ isomorphic to $M$. Let $a/b$ be a prime quotient of $M$. Then $f_a/f_b \in M[D]$ is isomorphic to $\mathcal{C}_< (X, 2)$. By a theorem of H. A. Priestley [6] this last lattice is isomorphic to $D$, hence we have the following

**Theorem 1.** Let $a/b$ be a prime quotient in a lattice $M$ and let $D$ be a bounded distributive lattice. Then there exists a lattice $M[D]$ containing the relative sublattice $M(aDb)$ such that the quotient $a/b$ of $M[D]$ is isomorphic to $D$, and $\Theta(a, b)/\omega$ is isomorphic to $\Theta(D)$.

For modular lattices this theorem was proved independently by R. Freese [3].

**Corollary 1.** If $\mathcal{X}$ is an equational class containing $M$, then $F_{\mathcal{X}}(M(aDb))$ exists.

Let $\tilde{M}(aDb)$ denote the sublattice of $M[D]$ generated by $M(aDb)$ (more precisely by $M(f_a D f_b)$). Then $\tilde{M}(aDb)$ has the following characterization

**Corollary 2.** $\tilde{M}(aDb)$ is the sublattice of $M[D]$ generated by

$$\{f_m \mid m \in M\} \cup \{h_d \mid d \in D\}$$

where

$$h_d(Q) = \begin{cases} a & \text{if } d \in Q, \\ b & \text{if } d \notin Q. \end{cases}$$
4. FINITE MODULAR LATTICES

If $M$ is a simple modular lattice then obviously $\tilde{M}(aDb) = M[D]$. The lattice given by the first diagram is also $\tilde{M}_3(a_1D0)$. Another characterization for $M_3[D]$ was given in [8], [9]: let $L$ be the poset of all triples $(x, y, z)$ $(x, y, z \in D)$ with the property $x \land y = y \land z = x \land z$, ordered by the rule: $(x, y, z) \leq (x', y', z')$ iff $x \leq x'$, $y \leq y'$, $z \leq z'$. The lattice operations of $L$ are:

$$(x_1, y_1, z_1) \land (x_2, y_2, z_2) = (x_1 \land x_2, y_1 \land y_2, z_1 \land z_2)$$

and

$$(x_1, y_1, z_1) \lor (x_2, y_2, z_2) = (x_1 \lor x_2) \lor [(y_1 \lor y_2) \land (z_1 \lor z_2)],$$

$$(y_1 \lor y_2) \land [(x_1 \lor x_2) \land (z_1 \lor z_2)],$$

$$(z_1 \lor z_2) \lor [(x_1 \lor x_2) \land (y_1 \lor y_2)].$$

Let $a_1, a_2, a_3$ denote the atoms of $M_3$, then the injections $a_1 \to (1, 0, 0)$, $a_2 \to (0, 1, 0)$, $a_3 \to (0, 0, 1)$, $d \to (d, 0, 0)$ $(d \in D)$ defines an embedding of $M(a_1D0)$ into $L$. In [8], [9] it was proved that the congruence lattices of $D$ and $L$ are isomorphic, moreover every congruence relation of $D$ can be extended to $L$. This yields that $L$ is a subdirect power of $M_3$. Let $P_i$ be a ultrafilter of $D$, then we denote by $\Theta[P_i]$ the extension of the congruence relation $\Theta[P_i]$ to $L$. Let $u$ be an element of $L$, then we can take the mapping $\tilde{u}: \tilde{u}: X \to M_3$ where $X$ is the set of ultrafilters of $D$ for which $\tilde{u}(P_i)$ is the image of $a_1$ by the natural homomorphism $\varphi: L \to L/\Theta[P_i]$ $(L/\Theta[P_i]$ is isomorphic to $M_3$). We get

**Proposition 6.** $L$ is isomorphic to $M_3[D]$.

The given representation of $M_3[3]$ is shown by the next diagram (the elements of $3$ are $0, 1, 1$).
Problem. Is it possible to give a similar characterization for $M[D]$ if $M$ is a finite simple complemented modular lattice?

Let $p_1, p_2, \ldots, p_n$ be the atoms of $M$. An element $(x_1, x_2, \ldots, x_n)$ of $D^n$ is called normal if $p_i \lor p_j \trianglerighteq p_k$ implies $x_i \land x_j = x_i \land x_k = x_j \land x_k$. Conjecture: $M[D]$ is the poset of all normal elements.

5. THE CHARACTERISATION OF $F_{\mathcal{M}}(M_3(0D_1))$.

($\mathcal{M}$ denotes the equational class of modular lattices.) In this section we give a simple proof for the main theorem of [5]. The proof is based on an interesting property of $M_3$.

Proposition 7. Let $M_3$ be a sublattice of a modular lattice $L$. If $f(x)$ and $g(x)$ are unary algebraic functions over $M_3$ then $f(0) = g(0)$ and $f(a_1) = g(a_1)$ imply $f(x) = g(x)$ for every $x \in L$ ($x \in a_1 / 0$).

Proof. The product of two unary algebraic functions $f_1$ and $f_2$ is defined by $f_1f_2(x) = f_1(f_2(x))$. Let us take the following special unary algebraic functions over $M_3$. $f_i = x \lor a_i$, $g_i(x) = x \land a_i$, $i(x) = x$ ($= x \lor 0 = = x \land 1$). Let $f$ be a unary algebraic function such that $f(0) \neq f(a_1)$.
Then $f$ is obviously the product of these special functions. Let $x$ be an element of $a_1/0$. Then for $u, v \in \{a_1, a_2, a_3\}$, $u \neq v$ we have:

(1) $f_u g_v f_u (x) = f_u (x)$ and $g_v f_u g_v (x) = g_v (x)$.

If $u, v, w$ are three distinct elements of $\{a_1, a_2, a_3\}$ then we prove:

(2) $f_u g_v f_w (x) = f_u g_w f_v (x)$ and $g_u f_v g_w = g_u f_w g_v$.

Take $f_u g_v f_w (x) \lor f_u g_w f_v (x)$. We can assume that $x \leq a_1$, since for other $x$ (2) is obviously satisfied. By the modularity we get

\[
f_u g_v f_w (x) \lor f_u g_w f_v (x) = \{(x \lor w) \land v \} \lor \{(x \lor v) \land w \} \lor u =
\]

\[
= \{(x \lor w) \land v \lor (x \lor v) \land w \} \lor u = \{(x \lor v) \land w \} \lor (u \lor x) =
\]

\[
= \{(x \lor v) \land w \} \lor u = f_v g_w f_u (x).
\]

By the symmetry of $v$ and $w$ we get (2). Using (1) and (2) a simple discussion proves our lemma.

**Theorem 2** (A. Mitschke and R. Wille [5]). *Let $N$ be a modular lattice and let $M_3(ODa_1)$ be a relative sublattice of $N$. The following statements are equivalent:*

(a) $N$ is generated by $M_3(ODa_1)$;

(b) $N$ is isomorphic to $E_{\#}(M_3(ODa_1))$;

(c) $N$ is isomorphic to the subdirect power of $M_3$ containing all quasi-real, continuous mappings of the Stone space $S(D)$ into the $T_0$-space $M_3$ with the subbasis $\{\{x\} | x \in M_3\}$. (For the notion see [4]).

**Proof of Theorem 1.** Let $M_3(ODa_1)$ be a relative sublattice of the modular lattice $L$, and denote by $N$ the sublattice generated by $M_3(ODa_1)$. We prove that $N$ is isomorphic to $N_0$, where $N_0$ is the lattice obtained from $D$ by taking all $(x, y, z)$ $(x, y, z \in D)$ with $x \land y = x \land z = y \land z$. Put $D' = \{x \in L | x = (d \lor a_3) \land a_2 \lor a_1, d \in D\}$. Then $D'$ is a distributive sublattice of $a_1 \lor a_2 / a_1$ isomorphic to $D$. 

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(i) For \( x \in N \) we set \( x_1 = x \wedge a_1 \), \( x_2 = ((x \wedge a_2) \vee a_3) \wedge a_1 \), 
\( x_3 = ((x \wedge a_3) \vee a_2) \wedge a_1 \). By the modularity of \( N \) we have \( x_1 \wedge x_2 = 
= (x \wedge a_1) \wedge ((x \wedge a_2) \vee a_3) \wedge a_1 = [(x \wedge a_2) \vee (x \wedge a_3)] \wedge a_1 = [(x \wedge a_2) \vee 
\vee (x \wedge a_3)] \wedge a_1 \). By the symmetry of \( a_2 \) and \( a_3 \) we get \( x_1 \wedge x_3 = 
= x_1 \wedge x_2 \). Finally, \( x_2 \wedge x_3 = [(x \wedge a_2) \vee a_3] \wedge a_1 = [(x \wedge a_2) \vee (x \wedge a_3)] \wedge a_1 \).
Thus \( x_1 \wedge x_2 = x_1 \wedge x_3 = 
= x_2 \wedge x_3 \), hence using the distributivity of \( D \):

\[
(3) \quad \text{if } x_1, x_2, x_3 \in D \text{ then } x_1 = (x_1 \vee x_2) \wedge (x_1 \vee x_3).
\]

(ii) Put \( x^{(1)} = a_1 \vee (x \wedge a_2) \vee (x \wedge a_3) \), \( x^{(2)} = a_2 \vee (x \wedge a_1) \vee 
\vee (x \wedge a_3) \), \( x^{(3)} = a_3 \vee (x \wedge a_1) \vee (x \wedge a_2) \).

From (1) we get

\[
(4) \quad a_1 \wedge x^{(2)} \wedge x^{(3)} = (a_1 \wedge x^{(2)}) \wedge (a_1 \wedge x^{(3)}) = 
= [a_1 \wedge (a_2 \vee (x \wedge a_1) \vee (x \wedge a_3))] \wedge [a_1 \wedge (a_3 \vee (x \wedge a_1)) \vee 
\vee (x \wedge a_2))] = [(x \wedge a_1) \vee (a_1 \wedge (a_2 \vee (x \wedge a_3)))] \wedge [(x \wedge a_1) \vee 
\vee (a_1 \wedge (a_3 \vee (x \wedge a_2))] = (x_1 \vee x_3) \wedge (x_1 \vee x_2) = x_1.
\]

Obviously \( (x \wedge a_2) \vee (x \wedge a_3) \leq x^{(2)} \) and \( x^{(3)} \), i.e. \( (x \wedge a_2) \vee 
\vee (x \wedge a_3) \leq x^{(2)} \wedge x^{(3)} \). Applying (2), from these inequalities, we get

\[
(5) \quad \text{if } x \in N \text{ and } x_i \in D \quad (i = 1, 2, 3) \quad \text{then } x = (x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3) 
\text{implies } x = (x \vee a_1) \wedge (x \vee a_2) \wedge (x \vee a_3).
\]

(iii) Let \( A \) be the set \( \{x; \; x = (x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3), 
x_1, x_2, x_3 \in D\} \). If \( x, y \in A \) then \( ((x \vee y) \wedge a_1) \vee ((x \vee y) \wedge a_3) \vee 
\vee ((x \vee y) \wedge a_3) = x \vee y \), i.e. \( A \) is a join semilattice.

(iv) By Proposition 7 if \( x \in A \) then

\[
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\]
\( x \lor a_1 = (x \land a_2) \lor (x \land a_3) \lor a_1, \ (x \lor a_2) \land a_3 \lor a_1, \ (x \lor a_3) \land a_2 \lor a_1 \) are in \( D' \) hence from (5) it follows that

If \( x \in N \) and

\[
x \lor a_1, ((x \lor a_2) \land a_3) \lor a_1, ((x \lor a_3) \land a_2) \lor a_1 \in D'
\]

(5')

then \( x = (x \lor a_1) \land (x \lor a_2) \land (x \lor a_3) \)

implies \( x = (x \land a_1) \lor (x \land a_2) \lor (x \land a_3) \).

\( A \) is therefore a meet semilattice too. \( A \) contains obviously the relative sublattice \( M_3(0Da_1) \). The representation \( x = x_1 \lor ((x_2 \lor a_3) \land a_2) \lor (x_3 \lor a_2) \land a_3 \) implies that \( A \) is generated by this partial lattice, hence \( A \) and \( N \) are isomorphic.

(v) Finally we prove that \( N \) and \( N_0 \) are isomorphic too. Let us take the correspondence \( x \rightarrow (x_1, x_2, x_3) \). But \( x_1 \land x_2 = x_1 \land x_3 = x_2 \land x_3 \), i.e. \( (x_1, x_2, x_3) \in N_0 \). Conversely let \( (u, \nu, \omega) \in N_0 \) and set \( x = u \lor [(\nu \lor a_3) \land a_2] \lor [(\omega \lor a_2) \land a_3] \). It is easy to verify that \( x_1 = u \), \( x_2 = \nu \), and \( x_3 = \omega \), the given correspondence is a one-to-one order preserving mapping. Thus \( N \cong N_0 \). The conditions (a) and (b) are also equivalent. For the proof of the equivalence of (a) and (c) we refer to [5].

Let \( L \) be a lattice from the equational class \( \mathcal{X} \), \( A \) prime quotient \( a/b \) of \( L \) is called \( \mathcal{X} \)-pure if for every extension \( M \in \mathcal{X} \) of \( L \) and for any two unary algebraic functions \( f(x) \), \( g(x) \) over \( L \) the conditions \( f(a) = g(a) \), \( f(b) = g(b) \) imply \( f(x) = g(x) \) for every \( x \in a/b, x \in M \). The finite lattice \( L \) is \( \mathcal{X} \)-pure if every prime quotient is \( \mathcal{X} \)-pure. By Proposition 7 \( M_3 \) is \( \mathcal{M} \)-pure.

Problem. Is it true that \( \tilde{M}(aDb) \cong \tilde{F}_{\mathcal{M}}(aDb) \) for \( M \in \mathcal{M} \) if and only if \( M \) is \( \mathcal{M} \)-pure?
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