On splitting modular lattices

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1. Introduction

A finite subdirectly irreducible algebra is splitting in a variety if there is a largest subvariety of this variety not containing it. The splitting lattices are those subdirectly irreducible lattices which are the bounded homomorphic images of finitely generated free lattices (R. McKenzie [2]). This result does not supply necessary and sufficient conditions for a splitting lattice in subvarieties of the variety of all lattices. Let $\mathcal{M}$ be the variety of all modular lattices. The description of splitting lattices in $\mathcal{M}$, i.e. of splitting modular lattices is an open problem. In this paper we give a necessary condition for a lattice $S$ to be splitting modular.

2. Preliminaries, result

We denote the five element modular non-distributive lattice by $M_5$; $M_5$ with an additional atom is called $M_4$, etc. We call an ordered five-tuple $(v, x, y, z, u)$ of elements from a modular lattice a diamond if these elements form a copy of $M_5$ with $v$ and $u$ as the bottom and the top elements, respectively. Two quotients $a/b$ and $c/d$ of a lattice $L$ are transposes if either $a = b \lor c$ and $d = b \lor c$ or $c = a \lor d$ and $b = a \lor d$. The quotient $a/b$ is said to be projective to $c/d$ — in symbol $a/b \approx c/d$ — if there exists a sequence of quotients $a/b = a_0/b_0, a_1/b_1, \ldots, a_n/b_n = c/d$ such that $a_0/b_0$ and $a_{k+1}/b_{k+1}$ are transposes for every $0 \leq k < n$. A sublattice $K$ of $L$ is called an isometric sublattice if a prime quotient in $K$ is a prime quotient in $L$. An element $a \in L$ is double-irreducible if it is join- and meet-irreducible. If $a$ is double-irreducible then $L_a = L \setminus \{a\}$ is a sublattice of $L$.

**Theorem.** Let $(v, x, y, z, u)$ be an isometric diamond of a splitting modular lattice $S$. If $y$ is double-irreducible then the quotients $x/v$ and $z/v$ are not projective in the sublattice $S_y = S \setminus \{y\}$.
This theorem implies

**Corollary 1** (A. Day, C. Herrmann and R. Wille [1]). \( M_4 \) is not splitting modular.

**Corollary 2.** The lattice represented by Fig. 1 is not splitting modular.

![Fig. 1.](image)

3. Function lattices

Let \( L \) be a lattice and let \( P \) be a partially ordered set. \( L^P \) denotes the lattice of all order-preserving maps of \( P \) to \( L \) partially ordered by \( f \leq g \) if and only if \( f(x) \leq g(x) \) for each \( x \in P \). \( L^P \) is called function lattice and this concept is a powerful tool by the construction given in this paper. If \( a \in L \) then \( \bar{a} \) denotes the corresponding constant mapping, i.e. \( \bar{a}(x) = a \) for each \( x \in P \). If \( a/b \) is a prime quotient of \( L \) then the corresponding quotient \( \bar{a}/\bar{b} \) of \( L^P \) is isomorphic to \( 2^P \), where \( 2 \) denotes the two element lattice. \( 2^P \) is a distributive lattice. Obviously \( L^P \) is a subdirect power of \( L \). The constant mappings form a sublattice of \( L^P \) which is isomorphic to \( L \); we can identify \( L \) with this sublattice.

Consider the chain \( N \) of non-negative integers, the corresponding ordinal is denoted by \( \omega \). Similarly, \( \omega^* \) is the ordinal corresponding to the chain of non-positive integers. Then using the well-known ordinal sum we get the ordinals \( \omega + 1 \), \( 1 + \omega^* \), \( \omega + 2 \) where \( \omega + 2 \) corresponds to

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0 < 1 < 2 < \ldots < d < \infty, \quad \omega + 1 \text{ corresponds to } 0 < 1 < \ldots < \infty, \quad \text{and } 1 + \omega^* \text{ corresponds to } 0 > -1 > -2 > \ldots > -\infty. \quad \text{Trivially } \omega + 1 \equiv 2^\omega \text{ and } \omega + 2 \equiv 2^{1+\omega^*}.
\]

Let \( D \) be a filter of \( 1 + \omega^* \) and let \( L \) be a finite lattice. If \( f \in L^D \) then there exists a \( -k \in D \) such that \( f(-k) \equiv f(n) \) for every \( -n \in D \). We define \( f \in L^{1+\omega^*} \) as follows: \( f(-n) = f(n) \) if \( -n \in D \) and \( f(t) = f(-k) \) if \( t \in D \). Then \( f+\bar{f} \) is obviously the canonical embedding of \( L^D \) into \( L^{1+\omega^*} \). If \( D \) is the filter \( \omega^* \) then we get an embedding \( L^{\omega^*} \rightarrow L^{1+\omega^*} \). The chain \( k = \{0, -1, \ldots, -k\} \) is a filter of \( 1 + \omega^* \) hence we get again an embedding \( L^k \rightarrow L^{1+\omega^*} \).

**Lemma 1.** Let \( L \) be a finite lattice. The ideal lattice \( I(L^{\omega^*}) \) is isomorphic to \( L^{1+\omega^*} \).

698
Proof. We have the canonical embedding \( f \to f \) of \( L^{0*} \) into \( L^{1+0*} \). Let \( g \in L^{1+0*} \) and take all \( f \in L^{0*} \) for which \( f \leq g \). All these \( f \)'s form an ideal \( I_g \) of \( L^{0*} \). It is easy to show that the correspondence \( g \to I_g \) is an isomorphism between \( L^{1+0*} \) and \( I(L^{0*}) \).

4. Gluing of lattices

Let \( A \) and \( B \) be two lattices with isomorphic sublattices \( C \cong C' \) where \( C \subseteq A \) and \( C' \subseteq B \). We assume that \( A \) and \( B \) are disjoint. The set-theoretical union \( L = A \cup B \) with \( C \) and \( C' \) identified can be made into a poset by defining \( x \equiv y \) if and only if one of the following conditions is satisfied:

(i) \( x \equiv y \) in \( A \) or in \( B \);

(ii) \( x \equiv c \) in \( A \) and \( c' \equiv y \) in \( B \) for some \( c \in A \) where \( c \) and \( c' \) are corresponding elements under the isomorphism \( C \cong C' \);

(iii) \( x \equiv c' \) in \( B \) and \( c \equiv y \) in \( A \) where \( c, c' \) are corresponding elements.

In general, \( L \) need not be a lattice. If \( L \) is a lattice then \( L \) is the lattice obtained by gluing together \( A \) and \( B \) by \( C \cong C' \). In the following we give a special condition for the sublattices \( C \) and \( C' \) such that \( L \) is a lattice.

A subchain \( C \) of a lattice \( L \) is called an \( m \)-subchain if the following conditions are satisfied:

1. If \( t \equiv c \) where \( t \in L \), \( c \in C \) then there exists a least \( t_i \in C \) such that \( t_i \equiv c \).

Similarly, if \( c \equiv t \) (\( c \in C \)) then we have a greatest \( t \in C \) such that \( c \equiv t \); \( t \);\( \equiv t \);

2. \( a \equiv b, \ a = b \) imply \( a = b \);

3. Let \( c_1 = c_2, \ c_1, \ c_2 \in C \). If \( c \equiv t \) then \( c_1 \vee t \equiv c_2 \vee t \) and dually \( c_1 \equiv t \) implies \( c_1 \wedge t \equiv c_2 \wedge t \);

4. \( r \equiv s, \ c \equiv t, \ r \equiv c \) imply that either \( r \vee c_1 \equiv s \vee c_1 \) or \( r \wedge c_1 \equiv s \wedge c_1 \);

5. \( \ell \equiv \ell, \ r \equiv s, \ c \equiv t \) imply \( c \equiv r \equiv s \) and dually.

A \( \{0, 1\} \)-subchain of a bounded lattice \( L \) is a subchain containing the 0 and 1 of \( L \).

Lemma 2. Let \( A \) and \( B \) be two modular lattices with isomorphic subchains \( C \cong C' \). Let \( C \) be an \( m \)-subchain of \( A \) and let \( C' \) be a \( \{0, 1\} \) \( m \)-subchain of \( B \). Then the poset \( L = A \cup B \) is a modular lattice.

Proof. First we show that \( L \) is a lattice. Take two elements \( a \in A, b \in B, a, b \in C \). Then we have a \( b \in C \), \( b \equiv b \). If \( a \equiv b \) then \( a \vee b = a \). If \( a \equiv b \) in \( A \) then by (1) we have \( a \equiv c \) such that \( a \leq a \equiv b \). Take the join \( a \vee b \) in \( B \) then this element is obviously the least upper bound of \( a \) and \( b \) in \( L \). If \( a \leq b \) then \( b \leq a \) hence the join \( a \vee b \) of \( a \) and \( b \) in \( A \) is the least upper bound of \( a \) and \( b \) in \( L \).
Similarly we can prove the existence of the greatest lower bound of $a$ and $b$, i.e. $L$ is a lattice.

Let us assume that $L$ is not modular, i.e. that $L$ contains a pentagon with the elements $o \prec s \prec r \prec i$, $o \prec t \prec i$. We distinguish several cases. $A$ and $B$ are modular lattices, hence $r, s, t \in A$ and similarly $r, s, t \in B$ is impossible.

(a) $r \in A$, $r \in B$, $s, t \in B$. Then $i = s \lor t$, $s, t \in B$ imply $i \in B$. From (1) we get the existence of the elements $\bar{r}, \bar{r} \in C$ for which $s \leq \bar{r} \prec t$. By the modularity of $B$ we get $s \lor (\bar{r} \land t) = \bar{r} \land (s \lor t) = \bar{r}$, i.e. $r \lor (\bar{r} \land t) = \bar{r}$, a contradiction to (3).

(b) $r, s \in A$, $t \in B$, $r, s, t \in C$. Then we have the following possibilities:

(ba) $a, i \notin B$. Using (1) we get $c_1 = i$ and $c_2 = t$ for which $i \succ c_1 \succ t > c_2 \succ o$. From (4) we conclude that either $r \lor c_1 > s \lor c_1$ or $r \land c_2 > s \land c_2$, which is a contradiction to the assumption that $a, r, s, t, i$ form a pentagon.

(bb) $a \notin B$, $i \in B$. Then we have $c_1 = \bar{r}$, $c_2 = \bar{r}$ for which $r \prec c_1 \equiv i$, $o \prec c_2 \prec t$. If $\bar{r} \succ \bar{s}$ then using the modularity of $B$ we get $s \lor (\bar{r} \land t) = \bar{r} \land i = \bar{r}$, $\bar{r} \lor t$, a contradiction to (3), i.e. $\bar{r} \equiv \bar{s}$. Then $c_2 \equiv r, s$, by (5) $r \lor c_2 > s \land c_2$, contradiction.

(ba) $a \notin B$. Let $c_1 = \bar{r}$, $c_2 = \bar{s}$, then $r \prec c_1 \equiv i$ and $s \succ c_2 \equiv a$. From (2) we get that either $\bar{r} \succ \bar{s}$ or $\bar{r} \succ \bar{a}$. Let us assume that $\bar{r} \succ \bar{s}$. Then by the modularity of $B$ we get $s \lor (\bar{r} \land t) = \bar{r} \land (s \lor t) = \bar{r} \land i = \bar{r}$, hence by (3) $\bar{r} \land t \in C$. But $C$ is a chain thus $\bar{r} \equiv \bar{r} \land t$, i.e. $\bar{r} \equiv t$, contradiction.

5. Proof of the theorem

Let $S$ be a finite subdirectly irreducible modular lattice with an isometric diamond $(r, x, y, z, u)$ such that $y$ is a double-irreducible element. Let us assume that the quotients $x/y$ and $z/y$ are projective in the sublattice $S_y = S \setminus \{y\}$. We have to prove that $S$ is not splitting modular.

First we take the function lattice $A = S^f_+ + w^*$. Then the quotient $u/x$ is a chain isomorphic to $\omega + 2$, say

$$x = x_0 < x_1 < x_2 < \ldots < x_d < x_\omega = u.$$  

Similarly $u/z$ is the following chain:

$$z = z_0 < z_1 < z_2 < \ldots < z_d < z_\omega = u.$$  

Let us take the elements: $w_0 = x_1 \land z_0$, $w_1 = x_2 \land z_1$, ..., $w_k = x_{k+1} \land z_k$, ..., $w_d = x_d \land z_d$, $w_\omega = u$. These elements form an $m$-subchain $C$ of $A$.

Let $B$ be a subdirect product of two copies of $\omega + 2$, containing all $(a, b) \in \omega + 2 \times \omega + 2$ for which $a \equiv b$. Then the elements $w'_0 = (0, 0), w'_1 = (1, 1), \ldots, w'_k = (k, k), \ldots, w'_d = (d, d), w'_\omega = (\omega, \omega)$ form a $\{0, 1\}$ $m$-subchain $C'$ of $B$ and $C$ is isomorphic to $C'$. The lattices $A$ and $B$ are illustrated in Fig. 2.
$B'$ will denote the principal ideal $(w'_0)$ of $B$.

Let $\tilde{M}$ be the lattice obtained by gluing together $A$ and $B$ identifying the corresponding elements under the isomorphism $C \cong C'$. By Lemma 2 $\tilde{M}$ is a modular lattice. We define some sublattices of $\tilde{M}$.

If we omit all elements $(k, \infty)$ $(k < \infty)$ from $\tilde{M}$ we get the sublattice $M$ of $\tilde{M}$. In other words, $M$ is the lattice obtained by gluing together $A$ and $B'$ identifying $w_0$ and $w'_0$, $w'_0$ and $w'_k$ $(k = 0, 1, \ldots)$.

The next step is to define the finite sublattices $M_k$ $(k = 0, 1, 2, \ldots)$ of $\tilde{M}$.

For a finite cardinal $k$ we define $A_k$ to be $S^k + 1$. Then by the canonical embedding defined in section 3, $A_k$ is a sublattice of $A$. The quotient $u/x$ is a $k + 2$ element chain

$$x = x_0 < x_1 < x_2 < \ldots < x_k < x_m = u;$$

hence the elements $w_0, w_1, \ldots, w_{k-1}$ are contained in $A_k$. Let $B_k$ be the principal ideal $(w'_k)$ of $B$. Then $M_k$ is the lattice obtained by gluing together $A_k$ and $B_k$ identifying the corresponding elements of the subchains $C_k = \{w_0, w_1, \ldots, w_{k-1}\}$ and $C'_k = \{w'_0, w'_1, \ldots, w'_{k-1}\}$. The corresponding diagram is given by Fig. 3.
Every $M_k$ is a sublattice of $M$, hence $M^* = \bigcup_{k=0}^\infty M_k$ is a sublattice of $M$.
By Lemma 1, the ideal lattice of $M^*$ is the lattice $M$, i.e. $I(M^*) = M$.

Lemma 3. $S$ is contained in the variety generated by the lattices $M_k$ for $1 \leq k < \infty$.

Proof. Let $\mathcal{K}$ be the variety generated by the lattices $M_k$ for $1 \leq k < \infty$.
Then $M^* = \bigcup_{k=0}^\infty M_k$ is in $\mathcal{K}$. This implies that the ideal lattice of $M^*$ is contained in $\mathcal{K}$, i.e. $M \in \mathcal{K}$. We will prove that $S$ is an epimorphic image of $M$. Therefore $S \in \mathcal{K}$.

Let $\theta$ be the congruence relation of $\omega + 2$ which has exactly two congruence classes, $\{0, 1, 2, \ldots\}$ and $\{d, \infty\}$. The factor lattice is the two element lattice.

Let $a/b$ be a prime quotient of $S$. Then there exists a natural isomorphism $e_{ab}: \omega + 2 \rightarrow a/b$, where $a/b$ is the corresponding quotient of $S$. Then $A = S_{\omega + \omega}^+$ has a congruence relation $\theta_A$ such that the factor lattice $A/\theta_A$ is isomorphic to $S$ and the restriction of $\theta_A$ to $a/b$ is the congruence relation which corresponds to $\theta$ by the isomorphism $e_{ab}$.

In the same way we get a congruence relation $\theta_B$ on $B'$ such that $\theta_B$ has the classes $\{w_d'\}$, $\{x; x \in B', x \equiv w_i' \text{ for some } i < d\}$ and $\{(k, d); k < d\}$. Let us take the chain $\{w_0, w_1, \ldots, w_d\} \subseteq A$. The restriction of $\theta_A$ to this chain has two classes: $\{w_0, w_1, \ldots\}$ and $\{w_d\}$. The restriction of $\theta_B$ to $\{w_0', w_1', \ldots, w_d'\} \subseteq B'$ has also the classes $\{w_0', w_1', \ldots, w_d'\}$ and $\{w_d'\}$. Let $\theta$ be the transitive extension of $\theta_A$ and $\theta_B$ to $M$. Then by the previous remark $\theta|A = \theta_A$ and $\theta|B = \theta_B$, $A/\theta_A \cong S$, $B'/\theta_B \cong 2$. Thus we get that $M/\theta$ is isomorphic to $S$, which proves our Lemma.

Let $\{M_k\}^*$ be the variety generated by $M_k$. The subdirectly irreducible lattices of a variety generated by a finite lattice $F$ are epimorphic images of sublattices of $F$. To prove that $S$ is not splitting we need to prove

Lemma 4. $S$ is not contained in the variety generated by $M_k$.

Proof. Let us take the quotient $u/v$ of $S$ and the corresponding quotient $u/v$ of $M_k$. It can be easily seen that $u/v$ is not an epimorphic image of a sublattice of $u/v$, using the assumption that $x/v$ and $y/v$ are projective in $S$. (See [1]). This involves that $M_k$ doesn’t contain a sublattice $T$ such that $S$ is an epimorphic image of $T$.

6. Planar lattices

Let $\mathcal{K}$ be a variety of lattices. A lattice $L$ in $\mathcal{K}$ is called finitely $\mathcal{K}$-projected if for any surjective $f: A \rightarrow L$ in $\mathcal{K}$ there is a finite sublattice of $A$ whose image under $f$ is $L$. In [3] the finitely projected planar modular lattices are characterized. From this characterization we get, using the concept of the diamond circle [4]:

702
Corollary 3. A subdirectly irreducible planar modular lattice $S$ is splitting modular if and only if $S$ does not contain a diamond circle or a sublattice isomorphic to $M_4$.

A planar modular lattice is 2-distributive. If $S$ is 2-distributive then the lattice $\overline{M}$ is again 2-distributive. Hence we have

Corollary 4. A subdirectly irreducible planar modular lattice $S$ is splitting in the variety of all 2-distributive lattices if and only if $S$ does not contain a diamond circle or a sublattice isomorphic to $M_4$.

Remark. The same proof gives the following generalization of our Theorem:

Let $(v, x, y, z, u)$ be an isometric diamond of a splitting modular lattice $S$ and let $t \in S$ be such that $u \wedge t = v$, $y$ is $\vee$-irreducible and $y \vee t$ is $\wedge$-irreducible. Then $S' = \{x \in S; x \not\subseteq y \cup t\}$ is a sublattice of $S$ and $x/v$, $z/v$ are not projective in this sublattice.

References


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703