The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice
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The congruence lattice of an arbitrary lattice is a distributive algebraic lattice, i.e. the ideal lattice of a distributive semilattice with 0. The converse of this statement is a long-standing conjecture of lattice theory. We prove the following:

Theorem. Let $L$ be the lattice of all ideals of a distributive lattice with 0. Then there exists a lattice $K$ such that $L$ is isomorphic to the congruence lattice of $K$.

The conjecture was first established for finite distributive lattices by R. P. Dilworth. Later, it was solved for the ideal lattice of relatively pseudo-complemented join-semilattices (E. T. SCHMIDT [4], [5]).

The first section of this paper reviews the definitions and gives the outline of the proof. The basic notion is the so-called distributive homomorphism of a semilattice (see [4]). The second section proves that for every distributive lattice $F$ with 0 there exists a generalized Boolean algebra $B$ — considered as a semilattice — and a distributive homomorphism of $B$ onto $F$. In the third section we prove the main result and in the last section we give some generalizations.

1. Preliminaries

Semilattice always means a join-semilattice in this paper. The compact elements of an algebraic lattice $L$ form a semilattice $L^c$ with 0, and $L$ is isomorphic to the ideal lattice of $L^c$. We denote by $\text{Con}(K)$ the congruence lattice of the lattice $K$. The compact elements of $\text{Con}(K)$ are called compact congruence relations, these form the semilattice $\text{Con}^c(K)$.

Let $B$ be a sublattice of a lattice $K$. The connection between $\text{Con}^c(B)$ and $\text{Con}^c(K)$ is of course very loose. Let $\theta$ be a congruence relation of $B$.

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Then there exists a smallest congruence relation \( \theta \in \text{Con}^e(K) \) such that \( \theta |_B \equiv \theta \). It is easy to see that \( \theta_1 \lor \theta_2 = (\theta_1 \lor \theta_2) \phi \), i.e. the correspondence \( \theta \mapsto \theta \phi \) is a homomorphism of \( \text{Con}^e(B) \) into the semilattice \( \text{Con}^e(K) \). If this homomorphism is onto we call \( K \) a strong extension of \( B \) [1]; or we say that \( B \) is a strongly large sublattice. It is an important case if \( \theta |_B \equiv \theta \) holds, then we write \( \theta \) instead of \( \theta \phi \). \( \theta \) is called the extension of \( \theta \).

It is well known that in generalized Boolean lattices (i.e. relatively complemented distributive lattices with zero) there is a one-to-one correspondence between congruence relations and ideals and therefore if \( B \) denotes a generalized Boolean lattice then \( \text{Con}^e(B) \equiv B \). Let \( F \) be a distributive semilattice with 0. We would like to get a lattice \( K \) such that \( \text{Con}^e(K) = F \) holds. Therefore we start with a generalized Boolean lattice \( B \) which has a join-homomorphism onto \( F \) and we construct a strong extension \( K \) of \( B \) such that \( \theta \mapsto \theta \phi \) is the given join-homomorphism. The construction of a strong extension of this kind was developed in [4].

We will make a further assumption that \( B \) is a convex sublattice of \( K \). In this case the homomorphism \( \theta \mapsto \theta \phi \) has an additional property, formulated in the next proposition.

**Proposition 1.** Let \( B \) be a convex sublattice of \( K \) and let \( \theta = \phi \lor \psi \) where \( \theta, \phi, \psi \in \text{Con}^e(B) \). Then there exist \( \phi_1, \psi_1 \in \text{Con}^e(B) \) such that \( \phi_1 \lor \psi_1 = \theta \) and \( \phi_1 \equiv \phi \), \( \psi_1 \equiv \psi \).

**Proof.** \( \theta \) is a compact congruence relation of \( B \), hence \( \theta = \lor_{i=1}^n \theta(a_i, b_i) \), where \( a_i < b_i, a_i, b_i \in B \). From \( \theta \phi = \phi \lor \psi \phi \) we get \( \phi_i \equiv b_i(\phi \lor \psi \phi) \), \( i = 1, 2, \ldots, n \). We have therefore for every \( i \) a finite chain \( a_i = c_{0,i} < c_{1,i} < \ldots < c_{n,i} = b_i \) such that \( c_{j,i} \equiv \lor_{j=0}^{j=i} c_{j+1,i} \phi \) or \( c_{j,i} \equiv c_{j+1,i} \phi \). By the assumption, \( B \) is a convex sublattice, i.e \( c_{j,i} \in B \). Let \( \phi_i \) be the join of all principal congruences \( \theta(c_{j,i}, c_{j+1,i}) \in \text{Con}^e(B) \) with \( c_{j,i} \equiv c_{j+1,i} \phi \). In a similar way we get \( \psi_i \). Then \( \phi_i \equiv b_i(\phi_1 \lor \psi_1) \) for every \( i \), i.e. \( \theta = \phi_1 \lor \psi_1 \), and \( \phi_1 \equiv \phi \), \( \psi_1 \equiv \psi \).

This Proposition suggests the following.

**Definition 1.** Let \( S, T \) be two distributive semilattices. A homomorphism \( \varphi \) of \( S \) into \( T \) is called weak-distributive if \( \varphi(u) = \varphi(x \lor y) \) implies the existence of \( x_1, y_1 \in S \) such that \( x_1 \lor y_1 = u \), \( \varphi(x_1) \equiv \varphi(x) \), \( \varphi(y_1) \equiv \varphi(y) \) (see Figure 1).

![Figure 1](image-url)
The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice

The congruence relation induced by a weak-distributive homomorphism is called a weak-distributive congruence.

Let \( \varphi \) be a homomorphism of the semilattice \( S \) into the semilattice \( T \). The congruence relation of \( S \) induced by \( \varphi \) is denoted by \( \theta_{\varphi} \).

**Proposition 2.** Let \( S \) be a distributive semilattice. \( \varphi \colon S \to T \) is a weak-distributive homomorphism if and only if \( a \equiv b \lor c \ (\theta_{\varphi}), \ a \equiv b \lor c \) imply the existence of elements \( b_1 \equiv b, \ c_1 \equiv c \) such that \( b \equiv b_1 \ (\theta_{\varphi}), \ c \equiv c_1 \ (\theta_{\varphi}) \) and \( b_1 \lor c_1 = a \) (Figure 2).

**Proof.** Let us assume that \( \varphi \) is a weak-distributive homomorphism and let \( a \equiv b \lor c, \ \varphi(a) = \varphi(b \lor c) = \varphi(b) \lor \varphi(c) \), i.e. \( a \equiv b \lor c \ (\theta_{\varphi}) \). \( \varphi \) is weak-distributive, hence we have elements \( b_0, c_0 \in S \) such that \( b_0 \lor c_0 = a, \ \varphi(b_0) \equiv \varphi(b), \ \varphi(c_0) \equiv \varphi(c) \). Let \( b_1 = b \lor b_0, \ c_1 = c \lor c_0 \) then \( b_1 \lor c_1 = b \lor b_0 \lor c \lor c_0 = b \lor c \lor a = a \) and \( \varphi(b_1) = \varphi(b \lor b_0) = \varphi(b) \lor \varphi(b_0) = \varphi(b) \lor \varphi(b) = \varphi(b), \ i.e. \ b_1 \equiv b \ (\theta_{\varphi}) \). Similarly we get \( c_1 \equiv c \ (\theta_{\varphi}) \) which proves that \( \theta_{\varphi} \) satisfies the given property.

Let \( \theta_{\varphi} \) be a congruence relation with the property formulated in the Proposition. Let \( a[\theta_{\varphi}] = x[\theta_{\varphi}] \lor y[\theta_{\varphi}], \ i.e. \ a \equiv x \lor y \ (\theta_{\varphi}) \). Then \( a \lor x \lor y \equiv x \lor y \ (\theta_{\varphi}) \) and there exist \( x_1, y_1 \in S \) satisfying \( x_1 \lor y_1 = x \lor y \lor a, \ x \equiv x_1 \ (\theta_{\varphi}), \ y \equiv y_1 \ (\theta_{\varphi}) \). Therefore \( x_1 \lor y_1 \equiv a \), hence by the distributivity of \( S \) we get elements \( x_2, y_2 \) for which \( x_2 \equiv x_1, \ y_2 \equiv y_1 \) and \( x_2 \lor y_2 = a \). These elements satisfy \( \varphi(x_2) \equiv \varphi(x_1) \equiv \varphi(x) \), i.e. \( \varphi \) is weak-distributive.

It is easy to give an example for a semilattice \( S \) and \( a, b \in S \) such that there is no smallest weak-distributive congruence satisfying \( a \equiv b \ (\theta) \), i.e. the principal weak-distributive congruence does not exist. We follow another way to define a special weak-distributive congruence which plays the role of the principal congruence. The principal congruences of a semilattice have the property that every congruence class contains a maximal element.

**Definition 2.** [4] A congruence relation \( \theta \) of a semilattice is called monomial if every \( \theta \)-class has a maximal element.
The monomial congruence are special meet-representable congruences. Every congruence relation of a semilattice is the join of principal congruence relations therefore it is natural to introduce the following notion.

**Definition 3.** [4] A congruence relation \( \theta \) of a semilattice is called *distributive* if \( \theta \) is the join of weak-distributive monomial congruences. A homomorphism \( \varphi: S \rightarrow T \) is distributive iff the congruence relation \( \theta \) induced by \( \varphi \) is distributive.

**Remark.** It is easy to prove that the join of weak-distributive congruences is weak-distributive. The basic properties of distributive congruences are listed in [6].

If \( (B; \vee, \wedge) \) is a generalized Boolean lattice, then the semilattice \( (B; \vee) \) will be called a generalized Boolean semilattice.

For the solution of the characterization problem of congruence lattices of lattices it is enough to solve the following two problems:

**Problem 1.** Let \( B \) be a generalized Boolean semilattice and let \( \theta \) be a distributive congruence of \( B \). Does there exist a lattice \( K \) satisfying \( \text{Con}^* (K) \cong B/\theta \)? Does there exist a strong extension of \( B \) satisfying the same property?

This problem was solved positively in [4]. In section 3 we give the sketch of the proof.

**Problem 2.** Let \( F \) be a distributive semilattice with \( 0 \). Does there exist a generalized Boolean semilattice \( B \) and a distributive congruence \( \theta \) of \( B \) such that \( F \) is isomorphic to \( B/\theta \)?

This problem is open. We solve this problem if \( F \) is a lattice, i.e. we prove the following.

**Theorem 1.** Let \( F \) be a distributive lattice with \( 0 \). Then there exist a generalized Boolean semilattice \( B \) and a distributive congruence \( \theta \) of \( B \) such that \( F \cong B/\theta \).

The proof of this theorem will be given in the next sections. We present here the basic idea of the proof.

Let \( F \) be a semilattice, \( a, b \in F \). The pseudocomplement \( a \ast b \) of \( a \) relative to \( b \) is an element \( a \ast b \in F \) satisfying \( a \vee x \leq b \) iff \( x \leq a \ast b \). If \( a \ast b \) exists for all \( a, b \in F \) then \( F \) is a relatively pseudocomplemented semilattice. (In the literature the pseudocomplement is usually defined in meet-semilattices.)

Let \( F \) be a relatively pseudocomplemented lattice (i.e. the join-semilattice \( F^\vee \) is relatively pseudocomplemented). The proof of Theorem 1 in this case is quite easy. Let \( B \) be the Boolean lattice \( R \)-generated by \( F \). (See [2], p. 87.) Then for every \( x \in B \) there exists a smallest \( \overline{x} \in F \) satisfying \( x \leq \overline{x} \). The mapping \( x \rightarrow \overline{x} \) is a distributive homomorphism of \( B \) onto \( F \). The congruence relation induced by this mapping is
monomial. The converse of this statement is true: if \( \theta \) is a monomial distributive
congruence of \( B \) then \( B/\theta \) is a relatively pseudocomplemented lattice.

If \( F \) is a relatively pseudocomplemented semilattice then this construction does not work. In this case we consider for every \( a \in F, a \neq 0 \) the skeleton of \( (a) \), i.e. \( S(a) = \{ x*a; x \equiv a \} \) ([2], p. 112). \( S(a) \) is a Boolean lattice. Consider the lower discrete direct product \( \prod_a (S(a); a \in F, a \neq 0) \), i.e. the sublattice of the direct product
\( \prod S(a) \) of those sequences \( t \) for which \( t(a) = 0 \) for all but finitely many \( a \in F \). This is a generalized Boolean lattice \( B \), and it is easy to show that \( B \) has a distributive congruence \( \theta \) satisfying \( B/\theta \cong F \) (see [4]).

To prove Theorem 1 we generalize the notion of the skeleton. Let \( \varphi \) be the
identity \( \varphi : S(1) \rightarrow F \). If \( B \) denotes \( S(1) \) and \( 0, 1 \in B \) then this \( \varphi \) obviously has the following properties:

1. \( \varphi \) is a \( \{0, 1\} \)-homomorphism of the Boolean semilattice \( B \) into the semilattice \( F \),
2. if \( \varphi (1) = x \lor y \) in \( F \) then there exist \( x_1, y_1 \in B \) such that \( x_1 \lor y_1 = 1 \), \( \varphi (x_1) \equiv x \), \( \varphi (y_1) \equiv y \).

(1) follows from the property that \( S(a) \) is a subsemilattice of \( F \), and (2) is obvious if we take \( x_1 = y \ast 1, y_1 = x_1 \ast 1 \).

Definition 4. Let \( F \) be a distributive semilattice with 0, 1 \( \in F \) and let \( B \)
be a Boolean semilattice with unit element \( 1 \) and zero element \( 0 \). \( B \) is called a pre-
skeleton of \( F \) if there exists a mapping \( \varphi \) of \( B \) into \( F \) such that conditions (1) and (2) are satisfied.

Condition (2) is related to the distributivity of \( \varphi \); if (2) is satisfied for every
\( a \in B \) (instead of \( 1 \)) and \( \varphi \) is onto then we get that \( \varphi \) is distributive.

2. The pre-skeleton

To prove Theorem 1 we shall show that every bounded distributive lattice has a
pre-skeleton. First we verify some simple well-known properties of free Boolean
algebras. The free Boolean algebra \( B \) generated by the set \( G \) is denoted by \( F(G) \). If
\( |G| = m \) we shall write \( F(m) \) for \( F(G) \). \( 1 \) denotes the unit element of \( F(G) \). Let \( G' = \{ x'; x \in G \} \) (\( x' \) denotes the complement of \( x \)) and \( G_1 = G \cup G' \). For \( g \in G \), \( g^* \) is
either \( g \) or \( g' \). Let \( k \) be a natural number. We consider the subset \( G_k \) of \( B \) defined by
\( G_0 = \{ 1 \} \) and \( G_k = \{ x; x \in B, x \neq 0, x = g_1^* \land \ldots \land g_k^* \}, \) where \( g_1, \ldots, g_k \) are different
elements of \( G \). From these sets \( G_k \) we get \( \mathcal{H} = \bigcup_{i=0}^{\infty} G_i \). If \( |G| = n \) is a natural
number then \( G_n \) is the set of atoms of \( F(n) \) and each \( a \in F(n), a \neq 0 \) has a unique
representation as a join of elements of \( G_n \). If \( G \) is infinite we have no atoms, therefore we must take the whole set \( \mathcal{H} \), which is of course a relative sublattice of \( B \).
The most important properties of \( \mathcal{H} \) are collected in the following definition.

**Definition 5.** A relative sublattice \( \mathcal{H} \) of a Boolean algebra \( B \) is called a join-base iff the following conditions are satisfied:

(i) \( 0 \notin \mathcal{H} \) and \( 1 \in \mathcal{H} \).

(ii) Each \( a \in B, a \neq 0 \) has a representation as a join of elements of \( \mathcal{H} \).

(iii) There is a dimension function \( \delta \) from \( \mathcal{H} \) onto an ideal of the chain of non-negative integers such that \( \delta(1) = 0 \) and \( x < y \) in \( \mathcal{H} \) if and only if \( x \equiv y \) and \( \delta(x) = \delta(y) + 1 \). The set of all \( x \in \mathcal{H} \) with \( \delta(x) = i \) is denoted by \( \mathcal{H}_i \).

(iv) For every finite subset \( U = \{ u_1, \ldots, u_n \} \) of \( B \) there exists an \( i \in \mathbb{N} \) such that each \( \mathcal{H}_k \) (\( k \equiv i \)) has a finite subset \( A_k(U) \) with the property that each \( u \in U \) has a unique join representation as a join of elements of \( A_k(U) \).

(v) If \( a \lor b \notin 0 \in B, a, b \in \mathcal{H} \) then \( a \lor b \in \mathcal{H} \); if \( a \lor b \) exists in \( \mathcal{H} \) and \( a, b \) are incomparable then \( a, b \in \mathcal{H}_i, a \lor b \in \mathcal{H}_i \) for some \( i \in \mathbb{N} \). Assume, that there exists an \( a_0 \in \mathcal{H}_{i-1} \), \( a_0 \neq a \lor b \), \( a_0 \supset a \), then there is a \( b_0 \in \mathcal{H}_{i-1} \) such that \( a_0 \lor b_0 \) exists and \( a_0 \lor (a \lor b) = a, b_0 \lor (a \lor b) = b \).

Let \( \mathcal{H} \) be a join-base of a Boolean semilattice \( B \) and let \( f: \mathcal{H} \rightarrow L \) be a homomorphism into a distributive lattice (i.e., \( f(a \lor b) = f(a) \lor f(b) \) whenever \( a \lor b \) exists, and the same for \( \lor \)). We want to extend \( f \) to a homomorphism \( \varphi: B \rightarrow L \) (i.e., \( \varphi \) will be a join-homomorphism of the Boolean algebra \( B \)). Let \( a = h_1 \lor \ldots \lor h_n \) where \( h_i \in \mathcal{H} \). The only way to define \( \varphi \) is the following: \( \varphi(a) = f(h_1) \lor \ldots \lor f(h_n) \).

Condition (iv) yields that this definition is unique and (ii) implies that \( \varphi \) maps \( B \) into \( L \).

**Definition 6.** The homomorphism \( \varphi \) of the Boolean semilattice into \( L \) is called an \( L \)-valued homomorphism of \( B \) induced by \( f \).

To prove Theorem 1 we need the definition of free \( \{0, 1\} \)-distributive product (see G. GRÄTZER [2], p. 106).

**Definition 7.** Let \( D \) be the class of all bounded distributive lattices and let \( L_i, i \in I \) be lattices in \( D \). A lattice \( L \) in \( D \) is called a free \( \{0, 1\} \)-distributive product of the \( L_i, i \in I \), iff every \( L_i \) has an embedding \( \varepsilon_i \) into \( L \) such that

(i) \( L \) is generated by \( \bigcup (\varepsilon_i L; i \in I) \).

(ii) If \( K \) is any lattice in \( D \) and \( \varphi_i \) is a \( \{0, 1\} \)-homomorphism of \( L_i \) into \( K \) for \( i \in I \), then there exists a \( \{0, 1\} \)-homomorphism \( \varphi \) of \( L \) into \( K \) satisfying \( \varphi_i = \varphi \varepsilon_i \) for all \( i \).

The free \( \{0, 1\} \)-distributive product is denoted by \( \Pi^*(A_i; i \in I) \) or by \( A \ast B \). The lower discrete direct product is denoted by \( \Pi_\downarrow (A_i; i \in I) \) and finally if \( A_i \) are lattices with unit element then \( \Pi^\uparrow (A_i; i \in I) \) is the upper discrete direct product,
i.e. the sublattice of the direct product \( \prod A_i \) of those sequences \( t \) for which \( t(a) = 1 \) for all but finitely many \( a \).

**Lemma 1.** Let \( L \) be a bounded distributive lattice and let \( A_i \) \((i \in I)\) be Boolean semilattices. If \( \varphi_i: A_i \to L \) \((i \in I)\) are \( L \)-valued \( \{0,1\} \)-homomorphisms generated by \( f_i: \mathcal{H}^i \to L \) then the free \( \{0,1\} \)-distributive product \( \Pi^*A_i \) has a join-base \( \mathcal{H} \) and a homomorphism \( f: \mathcal{H} \to L \) such that \( \mathcal{H} \cap A_i = \mathcal{H}^i \) for each \( i \in I \). There exists an \( L \)-valued homomorphism \( \varphi \) of \( \Pi^*A_i \) generated by \( f \) satisfying \( \varphi_i = \varphi \mathcal{H}^i \).

**Proof.** Let \( \mathcal{H} \) be the set of all those elements \( h \not= 0 \) of \( \Pi^*A_i \) which have a finite meet-representation as a meet of elements from \( \bigvee \mathcal{H}^i \). (Then \( \mathcal{H} \) is isomorphic to the upper direct product \( \Pi^i \mathcal{H}^i \).) Obviously \( \mathcal{H}^i \subseteq \mathcal{H} \), \( \mathcal{H} = \mathcal{H} \cap A_i \). Let \( x = h_1 \land \ldots \land h_n \) where the \( h_i \in \mathcal{H}^i \) belong to different components, then this representation is unique. We have by (iii) the functions \( \delta_i: \mathcal{H}^i \to N \). Now let \( \delta: \mathcal{H} \to \mathcal{H} \) be defined by \( \delta(u) = \delta_1(h_1) + \ldots + \delta_n(h_n) \). It is easy to verify (iv) and (v). Assume that \( f_i: \mathcal{H}^i \to L \) are homomorphisms, then we can extend them as follows: \( f(u) = f_1(h_1) \land \ldots \land f_n(h_n) \). Hence \( x \leq y \Rightarrow (x, y \in \Pi^*A_i) \) implies \( f(x) \leq f(y) \). Let us assume that for incomparable \( b, c \in \mathcal{H} \), \( b \lor c \in \mathcal{H} \). Then by (v) there exist an \( i \) and \( b_0, c_0 \in \mathcal{H}^i \) such that \( b = b_0 \land (b \lor c) \) and \( c = c_0 \lor (b \lor c) \). Thus we get by the distributivity of \( L \) that \( f(b) \lor f(c) = f_1(b_0 \land (b \lor c)) \lor f_1(c_0 \lor (b \lor c)) = f_1(b_0 \lor f_1(c_0)) \lor f(b \lor c) \). But \( f_i: \mathcal{H}^i \to L \) is a homomorphism, hence \( f_1(b_0 \lor c_0) = f_1(b_0) \lor f_1(c_0) \). Obviously \( b_0 \lor c_0 \leq b \lor c \). Therefore \( f(b_0 \lor c_0) = f(b \lor c) \). This yields \( f(b) \lor f(c) = f(b \lor c) \), i.e. \( f \) is a homomorphism of \( \mathcal{H} \) into \( L \).

The free Boolean algebra on \( m \) generators is the free \( \{0,1\} \)-distributive product of \( m \) copies of the free Boolean algebra on one generator, i.e. if \( B_i \cong F(1), i \in I \) then \( F(m) \cong \Pi^*A_i \).

**Corollary.** If each \( B_i \cong F(1) \) has a \( \{0,1\} \)-homomorphism \( \varphi_i \) into the distributive lattice \( L \), then there exists an \( L \)-valued homomorphism \( \varphi \) of \( F(m) \) into \( L \) such that \( \varphi_i = \varphi \mathcal{H}^i \).

**Lemma 2.** Let \( L \) be a bounded distributive lattice. Then there exists a pre-skeleton \( B \) of \( L \).

**Proof.** First assume that \( B \) is a pre-skeleton and \( \psi: B_i \to B \) is a lattice homomorphism of the Boolean lattice \( B_i \) onto \( B \). Then it is easy to see that \( B_i \) is again a pre-skeleton and the corresponding join-homomorphism is \( \psi \psi(x) \). Therefore to prove our Lemma it is enough to take a free Boolean algebra generated by a "big" set.

We start with the set \( G \) with all pairs \((a, b)\) satisfying \( a, b \in L, a \lor b = 1, a, b \neq 0 \). Let \( G \) be a subset of \( G \) which is maximal with respect to the property: \((a, b) \in G \) iff \((b, a) \in G \).
In the free Boolean algebra $F(G)$ we define $(a, b)^\prime = (b, a)$, i.e. the complement of $(a, b)$ is $(b, a)$. The mapping $\varphi : F(G) \to L$ is defined as follows. For $(a, b) \in G_1$ we set $\varphi((a, b)) = a$ and let $\varphi(0) = 0$. Then $\varphi((a, b)) \lor \varphi((b, a)) = a \lor b = 1$, i.e. $\varphi$ is a $\{0, 1\}$-homomorphism of the semilattice $F((a, b))$ into $L$. Then by the Corollary to Lemma 1 there exists an extension $\varphi$ of these homomorphisms. Let $x' \lor y' = 1 = \varphi(I)$, $x, y \neq 1$, where $I$ denotes the unit element of $F(G)$. Take $x_1 = (x, y)$, $y_1 = (y, x) \in F(G)$. By the definition of $\varphi$ we have $\varphi(x_1) = x$, $\varphi(y_1) = y$, i.e. $F(G)$ is a pre-skeleton of $L$.

Example 1. As an illustration consider the lattice $L$ represented by Figure 3.

![Figure 3](image)

The set $G_1$ contains the pairs $(a, c)$, $(b, c)$, $(c, a)$, $(c, b)$ and for a generating set we can choose $G = \{(a, c), (b, c)\}$; then $B$ is the free Boolean algebra generated by two elements, i.e. $B \cong 2^4$. Figure 4 gives the join-homomorphism $\varphi$, in which the wavy line indicates congruence modulo $\theta = \text{Ker } \varphi$.

![Figure 4](image)
Remark. The set $G_1$ can be made into a poset as follows: $(x, y) \equiv (u, v)$ iff $x \equiv u$ and $y \equiv v$. We adjoin $0$ and $I$ and we take the Boolean algebra $B_1$ freely generated by this poset. $B_1$ is of course the homomorphic image of $B$ defined above. Sometimes it is easier to work with this “smaller” Boolean algebra (see Figure 5).

Example 2. Let $L$ be the lattice shown in Figure 6.

Let $N = \{0, 1, 2, \ldots \}$ be the set of all natural numbers. $B$ is the Boolean-algebra containing all finite and cofinite subsets of $N$. We define $(a_i, b) = \{x_i; x \equiv i\}$, $(b, a_i) = \{0, 1, \ldots, i-1\}$. Then $G = \{(a_i, b), (b, a_i); i = 0, 1, \ldots\}$ is a generating set. The corresponding join homomorphism is the following. Let $A$ be a subset of $N$ with the smallest element $f(A)$. If $A$ is finite then $\varphi(A) = b$ if $f(A) = 0$ and $\varphi(A) = a_{f(A)}$ if $f(A) > 0$. For an infinite $A$ we have $\varphi(A) = 1$ if $f(A) = 0$ and $\varphi(A) = a_{f(A)}$ if $f(A) > 0$. It is easy to see that $\varphi$ is a distributive homomorphism of $B$ onto $L$, which proves that $I(L) \cong L$ is the congruence lattice of a lattice. This is the simplest example to show that $\operatorname{Con}^e(K)$ need not to be relatively pseudocomplemented.

Lemma 3. Let $A_1$, $A_2$ be Boolean semilattices and let $\varphi_i; A_i \to L$ be $L$-valued $\{0\}$-homomorphisms generated by the homomorphisms $f_i; H_i \to L$ of the join-bases $H_i \subseteq A_i$ ($i = 1, 2$). Then $H = H_1 \cup H_2 \cup \{1\}$ is a join-base of $A_1 \times A_2$ and if $\varphi$ is the homomorphism generated by $f; H \to L$ then $\varphi = \varphi_1 + \varphi_2$.

Proof. The proof is obvious.

Remark. Lemma 3 is true for lower discrete direct product. In the infinite case this is a generalized Boolean algebra.

The basic idea of the proof of Theorem 1 can be illustrated by the following lattice (Figure 7).
Let \( a \) be an element of \( L \). Then \( [a] \) is a bounded distributive lattice. If \( B \) is a pre-skeleton of \( [a] \) then we write \( B=B(a); \ B(1) \) is a pre-skeleton of \( L \).

By Lemma 2 we have a homomorphism \( \varphi_a \) of the pre-skeleton \( B(1) \) onto the semilattice containing the elements \( \{1, a, b, c, d, 0\} \). Applying again Lemma 2 for the principal ideal \([a] \) we get the mapping \( \varphi_a \) of the pre-skeleton \( B(a) \) of \([a] \) onto \( \{a, d, e, b, f, 0\} \). Let \( x \) be an element of \( B(1) \) for which \( \varphi_a(x)=a \). \( B(1) \) is the direct product \( \langle x \rangle \times \langle x' \rangle \) where \( x' \) denotes the complement of \( x \). Take the free \( \{0, 1\} \)-distributive product \( C \) of \( \langle x \rangle \) and \( B(a) \). Let \( B \) be the Boolean semilattice \( C \times \langle x' \rangle \) then by Lemmas 1 and 3 \( \varphi_a \) and \( \varphi_x \) can be extended to a homomorphism \( \varphi : B \rightarrow L \) which is a distributive homomorphism onto \( L \).

We need the following

**Definition 8.** Let \( B \) be a Boolean semilattice and let \( L \) be a distributive lattice with 0. Let \( \varphi : B \rightarrow L \) be a 0-preserving distributive homomorphism. \((B, \varphi, L)\) is called a saturated triple if \( \varphi(u)=x \vee y \) implies the existence of \( x_1, y_1 \in B \) such that \( x_1 \vee y_1 = u \), \( \varphi(x_1) = x \), \( \varphi(y_1) = y \).

**Lemma 4.** If \((C, f, L), (D, g, L)\) are saturated triples then there exists a distributive homomorphism \( h : C \times D \rightarrow L \) such that \( h|_{C} = f \), \( h|_{D} = g \) and \((C \times D, h, L)\) is saturated.

**Proof.** For \((c, d) \in C \times D\) we define \( h((c, d)) = f(c) \vee g(d) \). Then \( h((c, 0)) = f(c) \vee 0 = f(c) \), \( h|_{C} = f \). Similarly \( h|_{D} = g \). Now
\[
\begin{align*}
h((a, b) \vee (c, d)) &= h((a \vee c, b \vee d)) = f(a \vee c) \vee g(b \vee d) = (f(a) \vee f(c)) \vee \\
&\quad \vee (g(b) \vee g(d)) = (f(a) \vee g(b)) \vee (f(c) \vee g(d)) = h((a, b)) \vee h((c, d))
\end{align*}
\]
which means that \( h \) is a homomorphism. We prove that \( h \) is distributive.

Let \( h(c, d) = f(c) \vee g(d) = x \vee y \) in \( L \). By the distributivity of \( L \) we get elements \( x_1, x_2, y_1, y_2 \in L \) such that \( x_1 \vee y_1 = f(c) \), \( x_2 \vee y_2 = g(d) \), \( x_1, x_2 \leq x \), \( y_1, y_2 \leq y \). Since \((C, f, L)\) is saturated, therefore we have \( c_1, c_2 \in C \) such that \( c_1 \vee c_2 = c \) and \( f(c_1) \leq x_1, f(c_2) \leq y_1 \). Similarly we get elements \( d_1, d_2 \in D \) with \( d_1 \vee d_2 = d, g(d_1) \leq x_2, g(d_2) \leq y_2 \). Set \( x = (c_1, d_2), y = (c_2, d_1) \). Then \( x \vee y = (c_1 \vee c_2, d_1 \vee d_2) = (c, d) \), \( h((c_1, d_1)) = f(c_1) \vee \\
\quad \vee g(d_1) = x_1, h(c_2, d_2) \leq y_2 \). This proves that \( h \) is weak-distributive. Let \( \theta = \text{Ker} f \), \( \Phi = \text{Ker} g \). Then \( \theta = \vee \theta_j, \Phi = \vee \Phi_j; \theta_j, \Phi_j \) are monomial distributive congruences. \( \theta_i \) resp. \( \Phi_j \) can be extended to \( C \times D, \theta_i \cup \Phi_j \) which are again monomial. It is easy to see that \( \text{Ker} h = (\theta_j \vee \Phi_j) \).

**Corollary.** Let \( C, D \) be two Boolean semilattices and \( f \) resp. \( g \) distributive homomorphisms of these Boolean semilattices into the distributive lattice \( L \). If \( f(C) \) resp. \( g(D) \) are ideals of \( L \) then there exists a distributive homomorphism \( h : C \times D \rightarrow L \) such that \( h|_{C} = f \), \( h|_{D} = g \).
Remark: In Lemma 4 $f$ and $g$ are not necessarily $L$-valuations induced by some join-bases.

Let $L$ be an arbitrary distributive lattice with 0. If $a \in L$, $a \neq 0$ the principal ideal $(a)$ is a bounded distributive lattice. Assume that for every $(a)$ we have a Boolean semilattice $B_a$ and a distributive homomorphism $\varphi_a$ of $B_a$ onto $(a)$. Consider the lower discrete direct product $B = \Pi_a (B_a | a \in L, a \neq 0)$. $B$ is a generalized Boolean semilattice. By Lemma 4 we have a distributive homomorphism $\varphi: B \rightarrow L$ which is onto. Consequently to prove Theorem 1 we can assume that $L$ is a bounded distributive lattice. By Lemma 2 we have a pre-skeleton $B(1)$ with a homomorphism $\varphi_1: B(1) \rightarrow L$ which satisfies (2). Let $u$ be an arbitrary non-zero element of the join-basis $H \subseteq B(1)$, $a = \varphi_1(u)$. The principal ideal $(a)$ of $L$ is a bounded distributive lattice, therefore we can apply again Lemma 2 to get a pre-skeleton $B(a)$ and a homomorphism $\varphi_a: B(a) \rightarrow (a)$ into $(a)$. If $u'$ denotes the complement of $u$ in $B(1)$ then $B = B(1)$ is the direct product $(u') \times (u)$. Take the free $\{0, 1\}$-distributive product $(u) \ast B(a)$ and finally the Boolean semilattice

$$B[I, u] = ((u) \ast B(a)) \times (u')$$

By Lemmas 1 and 3 we have a homomorphism $\varphi: B[I, u] \rightarrow L$, satisfying the following condition:

(1) if $r \in T = \{I, u\}$, $\varphi(r) = x \lor y$ then there exist $x_1, y_1 \in B[I, u]$ with $x_1 \lor y_1 = r$, $\varphi(x_1) \leq x$, $\varphi(y_1) \leq y$.

Using the same method for an element $v \in B \subseteq B[I, u]$ we get from $B[I, u]$ a Boolean algebra $B[I, u, v]$ satisfying (1) for the set $T = \{I, u, v\}$.


Proof. If $H$ denotes a join-base of $B$ and $x \in H$ then we shall write $H(x)$ for $H \cap \{x\}$. It is easy to show that $H(x) \cup H(x')$ is again a join-base and $L$-valuations generated by these join-bases coincide. If $u, v \in B$ then we have therefore a join-base $H(u \lor v) \cup H(u \lor v') \cup H(u' \land v) \cup H(u' \land v')$. Hence we get for $B[I, u, v]$ resp. $B[I, v, u]$ the following. Let $H_u$ resp. $H_v$ be a join base of $B(\varphi(u))$ resp. $B(\varphi(v))$; then $(H_u \cup H_v \cup H(1)) \cup (H_u \cup H_v \cup H(1)) \cup (H_u \cup H_v \cup H(1)) \cup (H_u \cup H_v \cup H(1))$ which proves the isomorphism.

Continuing this construction we get for arbitrary $u_1, u_2, ..., u_n \in B$ a Boolean semilattice $B[I, u_1, ..., u_n]$ and a homomorphism of this Boolean semilattice into $L$ such that condition (1) is satisfied for $T = \{I, u_1, ..., u_n\}$.

All these Boolean semilattices form a direct family. Let $C_1$ be the direct limit. Then $B(1) = C_1$ is a Boolean subalgebra of $C_1$ and we have $\varphi: C_1 \rightarrow L$ which satisfies (1) for all $x \in T = B(1)$. Then we start with $C_2$ and in the same way we get a Boolean semilattice $C_2$. Then $C_2$ is a Boolean subalgebra of $C_2$. Similarly, we get
$C_i$ ($i = 3, 4, \ldots$). These algebras $C_i$ form again a direct family. Let $\bar{B}$ be the direct limit. Let $\varphi: \bar{B} \to L$ be the corresponding homomorphism. Then $(B, \varphi, L)$ is saturated, hence $\varphi$ is a weak-distributive homomorphism into $L$.

**Lemma 6.** $\bar{B}$ has a join-base.

**Proof.** This is a trivial consequence of Lemmas 1 and 3.

**Lemma 7.** Let $\varphi: B \to L$ be a weak-distributive homomorphism of a Boolean semilattice $B$ generated by a homomorphism $f: H \to L$ of a join-base $H$. Then $\varphi$ is distributive.

**Proof.** Let $\theta$ be the congruence relation induced by $\varphi$. $H_k$ denotes the set of all $x \in H$ of dimension $k$. Take two elements $a, b \in B$, $a \succeq b$ satisfying $a = b$ ($\theta$). Then $a$ and $b$ have join-representations as joins of elements from some $H_k$, say $a = h_1 \vee \ldots \vee h_k \vee h_{k+1}$ and $b = h_1 \vee \ldots \vee h_k$. If $c = h_1 \vee \ldots \vee h_k$, $k \leq n$ and $d = h_1 \vee \ldots \vee h_n$, $i \leq k$ then $c \vee d = b$. By condition (iv) of Definition 5 we can assume that these representations of $a$, $b$, $c$, $d$ are unique. By the weak distributivity of $\theta$ we have elements $\tilde{c} \equiv c$, $\tilde{d} \equiv d$ such that $\tilde{c} \vee \tilde{d} = a$ and $\tilde{c} = \tilde{d} = \tilde{a}$ ($\theta$). For $\tilde{c}$, $\tilde{d}$ we have the following possibilities: (i) $\tilde{c} = c \vee h_{n+1}$, $\tilde{d} = d$; (ii) $\tilde{c} = c$, $\tilde{d} = d \vee h_{n+1}$; (iii) $\tilde{c} = c \vee h_{n+1}$, $\tilde{d} = d \vee h_{n+1}$.

We define a binary relation $\theta_{ab}$ on $B$ as follows: $x \equiv y$ ($\theta_{ab}$), $x \equiv y$ ($\theta$) and $y \equiv b$, $x \vee b = a$. Then the assumption that $\theta$ is induced by the join-base $H$ we get that each $\theta_{ab}$-class contains a maximal element. Let $\theta_{ab}^\vee$ be the smallest join congruence of $B$ satisfying $\theta_{ab}^\vee \equiv \theta_{ab}$. Then $u \equiv v$ ($\theta_{ab}^\vee$), $u \equiv v$ iff there exist $x \equiv y$, $x \equiv y$ ($\theta_{ab}$) such that $y \equiv v$ and $x \vee v = u$. Obviously $\theta_{ab}^\vee \equiv \theta$, $\vee \theta_{ab}^\vee = \theta$. The first part of the proof yields that $\theta_{ab}^\vee$ is distributive.

An element $a \in L$ is of finite order if there exists a sequence $a = x_0, x_1, x_2, \ldots, x_n$ such that $a = a \vee x_1 \leq a \vee x_1 \vee x_2 \leq a \vee x_1 \vee x_2 \vee \ldots \vee x_{n-1} \leq a \vee x_1 \vee \ldots \vee x_n = 1$ and $a \vee x_1 \vee \ldots \vee x_{i-1}$ is incomparable with $x_i$ ($i = 1, \ldots, n$). By the construction of $\varphi: B \to L$ the image of each $u \in \bar{B}$, $u \neq 0$ is the meet of elements of finite order. Now we have for every $a \in L$ a Boolean semilattice $B(a)$ and a distributive homomorphism $\varphi_a: B(a) \to (a)$ which maps $B(a)$ onto the set of all elements having a meet representation of elements of finite order in the lattice $(a)$. Then the triple $(B(a), \varphi_a, (a))$ is saturated. The lower discrete product of these Boolean semilattices $B$ has by Lemma 4 a distributive homomorphism onto $L$ which proves Theorem 1.
3. Construction of a strong extension

In this section we give the outline of the proof of the following theorem, which was proved in [4]. Combining Theorems 1 and 2 we get our main theorem.

**Theorem 2.** Let \( \theta \) be a distributive congruence of a generalized Boolean semilattice \( B \). The lattice of all ideals of \( B/\theta \) is the congruence lattice of a lattice.

We denote the five element modular non-distributive lattice by \( M_5 \); \( M_5 \) with an additional atom is called \( M_6 \), etc. If \( \alpha \) is an arbitrary cardinal number then \( M_\alpha \) is the modular lattice of length \( 2^\alpha \) with \( \alpha \) atoms.

Let \( M = \{ 0 < a, b, c < 1 \} \) be a lattice isomorphic to \( M_3 \) and let \( D \) be a bounded distributive lattice with zero element \( o \), and unit element \( i \). Identifying \( a \) with \( i \) and \( 0 \) with \( o \), we get a partial lattice \( M_5 = D \cup M_3 \) (Fig. 8), \( D \cap M_3 = \{ 0, a \} \) and \( D, M_3 \) are sublattices; \( d \lor b \) resp. \( d \lor c \) (\( d \in D \)) is defined iff \( d \in \{ 0, a \} \) (see MITSCHKE & WILLE [3]). There exists a modular lattice \( M_3[D] \) generated by \( pM_3 \) such that \( pM_3 \) is a relative sublattice of \( M_3[D] \). In [3] it was proved that there exists only one modular lattice with these properties, the modular lattice \( FM(pM_3) \) freely generated by \( pM_3 \). This lattice was introduced in [4] and has the following description.

![Diagram](image)

**Figure 8.**

An element \( (x, y, z) \in D \times D \times D \) is called *normal* if \( x \land y = x \land z = y \land z \). Let \( M_3[D] \) be the poset of all normal elements, then \( M_3[D] \) is a modular lattice. Let \( a = (i, 0, 0), \ b = (0, i, 0), \ c = (0, 0, i), \ 1 = (i, i, i), \ 0 = (0, 0, 0) \). Then these elements form a sublattice isomorphic to \( M_3 \). The set of all elements \( (x, 0, 0), \ (x \in D) \) form a sublattice isomorphic to \( D \). \( D \) is a strongly large sublattice of \( M_3[D] \), and every congruence relation \( \theta \in \text{Con} (D) \) can be extended to \( M_3[D] \), i.e. \( \text{Con} (D) \cong \text{Con} (M_3[D]) \). We can use the same construction for distributive lattices without unit element.

We prove Theorem 2 first for monomial congruences of Boolean semilattices i.e. for relatively pseudocomplemented lattices.

**Lemma 8.** Let \( \theta \) be a monomial distributive congruence of a generalized Boolean semilattice \( B \). Then there exists a lattice \( N \) such that \( \text{Con}^* (N) \cong B/\theta \).
Sketch of the proof. Consider $D = B$ and the corresponding lattice $M_\theta[B]$. We define a subset $N$ of $M_\theta[B]$ as follows:

\[ (**) \quad (x, y, z) \in M_\theta[B] \text{ belongs to } N \text{ iff } x \text{ is a maximal element of a } \theta\text{-class.} \]

Then $N$ is a lattice and $(x, 0, 0) \in N$ iff $x$ is a maximal element of $\theta$-class, i.e., the ideal $I$ generated by $(i, 0, 0)$ is isomorphic to $B/\theta$. $N$ is a strong extension of $I$, a congruence relation of $I$ has an extension to $N$ iff it has the form $\theta(I')$, where $I'$ is an ideal of $N$. Thus $\text{Con}^c(N) \cong B/\theta$, i.e., $\text{Con}(N) \cong I(B/\theta)$.

The ideal $J$ of $N$, generated by $(0, 0, i)$ is isomorphic to $B$. By the definition of $I$ and $J$ we have $I \cap J = 0$ (Fig. 9).

![Figure 9.](image1)

![Figure 10.](image2)

Let $\theta$ be an arbitrary distributive congruence relation of the generalized Boolean semilattice $B$. Then $\theta$ is the join of monomial distributive congruence relations, say $\theta = \vee_{\alpha} \theta_{\alpha}[x \in \Omega]$. We take first for every $\alpha$ the lattice $N_{\alpha}$ defined before. This $N_{\alpha}$ has two ideals $I_{\alpha} \cong B/\theta_{\alpha}$ and $J_{\alpha} \cong B$. Moreover $\text{Con}^c(N_{\alpha}) \cong B[\theta_{\alpha}]$.

On the other hand we consider the direct product $\Pi(B_{\alpha}[x \in \Omega]). M$ denotes the sublattice of the direct product of those normal sequences $t$ for which $\{t(\alpha)[x \in \Omega]\}$ is finite, i.e., the weak direct product is normal if $x, \beta, \gamma \in \Omega, \alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma$ imply $t(\alpha) \wedge t(\beta) = t(\alpha) \wedge t(\gamma) = t(\beta) \wedge t(\gamma)$. Let $J^* \alpha$ be the ideal of $M$ consisting of all $t$ for which $t(\beta) = 0$ if $\beta \neq \alpha$. Then $J^* \alpha \cong B$. $M$ is a strong extension of $J^* \alpha$ and $\text{Con}^c(M) \cong \text{Con}^c(J^* \alpha) \cong \text{Con}^c(B)$. Let $\overline{M}$ be the dual lattice of $M$. Then $J^* \alpha$ is a dual of $\overline{M}$. $J^* \alpha$ is a Boolean algebra, therefore we have a natural isomorphism $J^* \alpha \cong J^* (x \prec x')$. We use the Hall—Dilworth gluing construction for $\overline{M}$ and $N_{\alpha} (x \in \Omega)$, we identify for every $\alpha$ the dual ideal $J^* \alpha$ and the ideal $J_{\alpha}$. In this way we get a partial lattice $P$ (see Figure 10).

$\overline{M}$ and $N_{\alpha}$ are sublattices of $P$, and $P$ is a meet-semilattice. Let $F(P)$ be the free lattice generated by $P$. Then $\text{Con}^c(F(P)) \cong B[\theta]$. This proves Theorem 2.
4. Some remarks on the characterization problem

The key problem of the characterization of congruence lattices of lattices is to prove the existence of a pre-skeleton of a bounded distributive semilattice. We reformulate this problem.

Let $L$ be a bounded distributive semilattice. Let $F(G)$ be denote the free Boolean algebra generated by the set $G$. If $g_i \in G$ then the elements $0, g_i, g_i'$ form a Boolean subalgebra which is the free Boolean algebra $F(g_i)$ generated by $g_i$. We have remarked that $F(G)$ is the free \{0,1\}-distributive product of the Boolean algebras $F(g_i), g_i \in G$. Let us assume that every $F(g_i)$ has a \{0,1\}-homomorphism $\varphi_i$ into $L$. Does there exist a \{0,1\}-homomorphism $\varphi: F(G) \rightarrow L$ such that $\varphi|_{F(g_i)} = \varphi_i$? For finite $G$ the answer is yes, we have

**Proposition 3.** Let $B$ be a finite Boolean algebra. If $\varphi_1: B \rightarrow L$ and $\varphi_2: F(g) \rightarrow L$ are \{0,1\}-homomorphisms into $L$ then there exists a \{0,1\}-homomorphism $\varphi$ of the free \{0,1\}-distributive product $B \ast F(g)$ into $L$ such that $\varphi|B = \varphi_1$, $\varphi|_{F(g)} = \varphi_2$.

**Proof.** Let $p_1, p_2, \ldots, p_n$ denote the atoms of $B$. The atoms of the free product are $p_1 \wedge g, \ldots, p_n \wedge g$, $p_1 \vee g', \ldots, p_n \vee g'$. Then $g < p_1 \vee \ldots \vee p_n = I$ yields $\varphi_2(g) < \varphi_1(p_1) \vee \ldots \vee \varphi_1(p_n) = 1 \in F$. But $F$ is a distributive semilattice hence we have elements $a_1, a_2, \ldots, a_n \in F$ such that $\varphi_2(g) = a_1 \vee \ldots \vee a_n$, $a_i \equiv \varphi_1(p_i)$ ($i = 1, 2, \ldots, n$).

Similarly $g < p_1 \vee \ldots \vee p_n$ therefore we have elements $b_1, \ldots, b_n \in L$ satisfying $\varphi_2(g') = b_1 \vee \ldots \vee b_n$, $b_i \equiv \varphi_1(p_i)$.

On the other hand $p_i \equiv g \vee g'$ hence $\varphi_1(p_i) \equiv \varphi_2(g) \vee \varphi_2(g')$. Thus we get elements $u_i, v_i$ such that $\varphi_1(p_i) = u_i \vee v_i, u_i \equiv \varphi_2(g), v_i \equiv \varphi_2(g')$.

Define $\varphi(p_i \wedge g) = a_1 \wedge u_i$, $\varphi(p_i \vee g') = b_1 \vee v_i$. Every $u$ of $B \ast F(g)$ has a unique representation as a join of atoms, say $u = \vee g_i$. We define $\varphi(u) = \vee \varphi(g_i)$. This $\varphi$ is obviously a homomorphism. From $p_i = (p_i \wedge g) \vee (p_i \vee g')$ we get $\varphi(p_i) = (p_i \wedge g) \vee (p_i \vee g')$.

It is necessary to generalize Lemma 1 for distributive semilattice. Let $B$ be the free Boolean algebra $F(G)$. Then the join-base is $H = \bigcup_{i = 0}^{\infty} H_i \cup \{1\}$.

We have for every $g_i \in G$ a \{0,1\}-homomorphism $\varphi_i: F(g_i) = \{0, g_i, g_i', I\} \rightarrow L$, i.e. we have a mapping $H_i \rightarrow L$ and we want to get a \{0,1\}-homomorphism $\varphi: B \rightarrow L$ which is a common extension of each $\varphi_i$. To define such a $\varphi$ it is natural to use induction on $k$. If $x \in H_k$, then $x = g_i$ or $x = g_i'$ for some $g_i \in G$ and we have $\varphi(x) = \varphi_i(x)$. Using the method of Proposition 3 it is easy to define $\varphi(x)$ for all $x \in H_2$. How can we define $\varphi(x)$ for $x \in H_3$?
References


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