ON LOCALLY ORDER-POLYNOMIALY COMPLETE MODULAR LATTICES

E. T. SCHMIDT (Budapest)

1. Introduction

Let \( L \) be a lattice. \( F_k(L) = L^{(L^n)} \) is the set of all \( k \)-place functions on \( L \). If we define pointwise meet and join operation on \( F_k(L) \), then \( F_k(L) \) becomes a lattice. The elements of the sublattice \( P_k(L) \) of \( F_k(L) \) generated by the projections and the constant functions will be called \( k \)-place polynomial functions on \( L \). If \( f \in F_k(L) \) has the property that for every finite subset \( M \subseteq L^k \) there exists a \( p \in P_k(L) \) such that \( f \) and \( p \) coincide on \( M \), then \( p \) is called a local polynomial function. \( f \in F_k(L) \) is called order-preserving if \( a_i \leq b_i \), \( i = 1, \ldots, k \) implies \( f(a_1, \ldots, a_k) \leq f(b_1, \ldots, b_k) \). \( L \) is called (locally) order-polynomially complete iff every order-preserving function on \( L \) is a (local) polynomial function.

The first characterization of finite order-polynomially complete lattices was given in Wille [6]. For finite modular lattices he proved the following

**Theorem A.** A finite modular lattice \( L \) is (locally) order-polynomially complete if and only if \( L \) is simple and relatively complemented (i.e. an irreducible projective geometry).

This theorem suggests the following question: is every locally order-polynomially complete modular lattice relatively complemented? In [2], Fried proved that the answer is yes if \( L \) has locally finite length. Our main result is a construction of a locally order-polynomially complete modular lattice which is not relatively complemented. To prove that our example is locally order-polynomially complete we need the following useful result of Dorninger [1]:

**Theorem B.** A lattice \( L \) is locally order-polynomially complete if and only if

1. \( L \) is simple and
2. for all \( a, b \in L \), \( a \leq b \) and all 1-place polynomial functions \( p, q \) with \( p(b) = q(a) \) there exists a 1-place polynomial function \( r \) such that \( r(a) = p(a) \) and \( r(b) = q(b) \).

It is an easy consequence of this theorem that every simple, relatively complemented lattice is locally order-polynomially complete (for finite lattices see Wille [7]).

2. Preliminaries

By a 1-translation of a lattice \( L \) we mean a unary polynomial-function on \( L \) that is either the identity function \( id(x) \) or a constant function or is obtained from one of the two lattice operations by fixing one of the arguments. By a translation of \( L \) we mean a unary polynomial that is the composition of 1-translations. Therefore if \( t(x) \) is a translation but not a constant function then \( t(x) \) may be written in the

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form \( t(x) = \ldots ((x \land a_1) \lor a_2) \lor a_3 \ldots) \land a_n \) where each \( a_i \in L \) or is the empty symbol. We say that a translation is a unary polynomial function of degree 1. If \( f(x) \) and \( g(x) \) are unary polynomial functions of degree \( n \) resp. \( m \), then the degree of the functions \( f(x) \land g(x) \) and \( f(x) \lor g(x) \) is \( n + m \).

If two intervals \([a, b]\) and \([c, d]\) in a lattice are such that \( a = b \land c \) and \( d = b \lor c \), then each is said to transpose (perspective) of the other. \([a, b]\) and \([c, d]\) are said to be projective if there exists intervals \([a, b] = [x_0, y_0], [x_1, y_1], \ldots, [x_n, y_n] = [c, d] \) such that any two successive intervals are transposes of each other. In a modular lattice, any two projective intervals are isomorphic. A well-known property of modular lattices (see [3], p. 133) is expressed in the following

**Lemma 1.** Let \( t(x) \) be a translation of a modular lattice \( L \) and let \( a, b \in L \), \( a < b \) such that \( t(a) \neq t(b) \). Then there exists a proper subinterval \([a', b']\) of \([a, b]\) such that \([a', b'] \land (t(a), t(b))\) are projective.

Let \( D \) be an arbitrary distributive lattice with 0 and 1. Take the subposet of \( D^3 \) consisting of all ordered triples \((a, b, c)\) such that \( a \land b = a \land c = b \land c \). This poset is a modular lattice \( M_3[D] \) (see Schmidt [4]). The elements \( i = (1, 1, 1), u = (1, 0, 0), v = (0, 1, 0) \), \( w = (0, 0, 1) \) and \( o = (0, 0, 0) \) form a diamond, \( M_5 \). The interval \([0, u]\), i.e. the ideal \( (u) \) is isomorphic to \( D \). Similarly, \( (v) = (w) = D \).

Let us take two bounded lattices \( L_1 \) and \( L_2 \). Suppose that \( L_1 \) has a principal dual ideal \( J_1 \), \( L_2 \) has a principal ideal \( J_2 \) and \( J_1 \cong J_2 \). Let \( \varphi: x \mapsto x' \) denote this isomorphism. We can construct a lattice \( L \) as follows: \( L \) is the set of all \( x \in L_1 \) and \( x \in L_2 \); we identify \( x \) with \( x' \) for all \( x \in J_1 \). \( x \equiv y \) has unchanged meaning if \( x, y \in L_1 \) or \( x, y \in L_2 \) and \( x \equiv y \). \( x, y \in J_1 \) if and only if \( x \in L_1 \), \( y \in L_2 \) and there exists a \( z \in J \) such that \( x \sim z \) in \( L_1 \), and \( z \sim y \) in \( L_2 \). It is easy to see that \( L \) is a modular lattice if so are \( L_1 \) and \( L_2 \). This is the so-called Hall—Dilworth construction.

**3. Modular lattices of finite length**

By Theorem B, a locally order-polynomially complete lattice is simple. A direct proof is the following (see Wille [7]): let \( \Theta \) be a non trivial congruence relation of \( L \) and \( a, b, c \in L \) such that \( a \equiv b \), \( c \equiv d \), \((a, b) \in \Theta \) and \((c, d) \in \Theta \). We define a mapping \( f: L \rightarrow L \) by

\[
 f(x) := \begin{cases} 
  c, & \text{if } a \equiv x \\
  d, & \text{if } a \not\equiv x 
\end{cases}
\]

then \( f \) is an order-preserving function and it cannot be a local polynomial function, namely \((a, b) \in \Theta \) but \((c, d) = (f(a), f(b)) \in \Theta \).

**Proposition.** Let \( p(x) \) be a polynomial function on a modular lattice \( L \) of locally finite length. If the interval \([u, v]\) is a complemented sublattice then \([p(u), p(v)]\) is complemented, too.

**Proof.** We prove this statement by induction on the degree of \( p(x) \). (The degree of \( p(x) \) is the number of occurrences of the variable \( x \); the constant function has degree 0.) If \( p(x) \) has degree 1 (i.e. it is a translation) then by Lemma 1 \([p(u), p(v)]\) is isomorphic to a subinterval of a complemented modular lattice \([u, v]\), hence \([p(u), p(v)]\) is complemented. Assume that the assertion is proved for polynomials

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of degree \( \leq n \) \((n > 1)\), and let \( p(x) \) be a polynomial function of degree \( n \). Then \( p(x) \) has one of the following two decompositions, \( p(x) = q(x) \lor r(x) \) or \( p(x) = q(x) \land r(x) \) where the degree of \( q \) and \( r \) is less than \( n \). Denote \( p(u) \) by \( s \) then \( q'(x) = q(x) \lor s \) is a polynomial function and has the same degree as \( q \). Then by our assumption \( [q'(u), q'(v)] \) is complemented. On the other hand \( q'(u) = q(u) \lor s = q(u) \lor p(u) = p(u) \) and \( q'(v) = q(v) \lor s = q(v) \lor p(u) \), hence \( [p(u), p(u) \lor q(v)] \) is a complemented interval of finite length. Then the unit element (i.e. \( p(u) \lor q(v) \)) is the join of atoms of this interval. Similarly, \( p(u) \lor r(v) \) is the join of atoms in \( [p(u), p(u) \lor q(v)] \). We claim that \( p(v) = (p(u) \lor q(v)) \lor (p(u) \lor r(v)) \) is the join of atoms in \( [p(u), p(v)] \), hence this interval is complemented.

**Theorem C** (Fried [2]). Let \( L \) be a modular lattice of locally finite length. \( L \) is locally order-polynomially complete iff each interval of \( L \) is an irreducible projective geometry.

**Proof.** If each interval is an irreducible projective geometry then \( L \) is a relatively complemented, simple lattice; hence \( L \) is locally order-polynomially complete.

Conversely, let us assume that \( L \) is locally order-polynomially complete. If \( L \) is not relatively complemented then \( L \) contains a triple \( a, b, c \) such that \( a \prec b \prec c \) and \( b \) has no relative complement in \( [a, c] \). Let \( u \prec v \) be any two elements of \( L \) and define

\[
 f(x) := \begin{cases} 
 a, & \text{if } x \equiv u \\
 c, & \text{if } x \equiv u.
\end{cases}
\]

Then \( f(x) \) is an order-preserving function and by the Proposition, \( f \) cannot be a local polynomial function. Let \( [a, b] \) be an interval of \( L \) and let \( c_1, c_2 \) \((c_1 \neq c_2)\) be two atoms of \([a, b]\). To prove that \([a, b]\) is an irreducible projective geometry, it is enough to show that there exists an atom \( d \) of \([a, b]\) such that \( d \neq c_1, c_2 \) and \( d \equiv c_1 \lor c_2 \).

Since \( L \) is a simple modular lattice, the intervals \([a, c_1]\) and \([a, c_2]\) are projective in \( L \), hence they are projective in some interval \([a, b]\), where \( a \equiv a \prec b \equiv b \). The interval \([a, b]\) is again a complemented modular lattice of finite length, therefore \([a, b]\) is the direct product of irreducible projective geometries ([3], p. 212). The projectivity of \([a, c_1]\) and \([a, c_2]\) in \([a, b]\) yields that these intervals belong to the same irreducible component, i.e. \([a, c_1 \lor c_2]\) is a subinterval of an irreducible projective geometry, therefore there exists a \( d \) with \( a \prec d \prec c_1 \lor c_2 \).

**4. The construction**

We prove the following:

**Theorem D.** There exists a locally order-polynomially complete modular lattice which is not relatively complemented.

Let \( Q \) be the interval \([0, 1]\) of rational numbers. First, we define two unary operations on \( Q \):

\[
 f(x) := \begin{cases} 
 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\
 1, & \text{if } \frac{1}{2} < x
\end{cases}
\]

\[
 g(x) := \begin{cases} 
 2x - 1, & \text{if } \frac{1}{2} \leq x \leq 1 \\
 0, & \text{if } x < \frac{1}{2}.
\end{cases}
\]

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and we consider the unary algebra $Q = \langle Q; f, g \rangle$. Let $\text{id}$ be the identity map on $Q$ and define $f^0 = g^0 = \text{id}$. Then apart from the constant maps the polynomial functions on $Q$ are of the form $p(x) = g^k f^i g^k \ldots g^k f^k$, where $k_i \equiv 0$, $l_i \equiv 0$.

**Lemma 2.** To each $a, b \in Q$, $a \prec b$ there exists a 1-place polynomial function $p(x)$ on $Q$ such that $p(a) = 0$ and $p(b) = 1$.

**Proof.** If $0 \equiv a \prec b \equiv 1$ then for suitable $k$ and $n$ ($n \equiv 1, \ k = 0, \ldots, 2^n - 1$) \[ a \equiv \frac{k}{2^n} \prec \frac{k + 1}{2^n} \equiv b. \] Therefore if we have a $p(x)$ such that $p\left(\frac{k}{2^n}\right) = 0$ and $p\left(\frac{k + 1}{2^n}\right) = 1$ then by the order-preserving property of polynomials we get $p(a) = 0$, $p(b) = 1$, i.e. we can assume that $a = \frac{k}{2^n}$ and $b = \frac{k + 1}{2^n}$. We prove the lemma by induction on $n$. If $n = 1$ then we have:

\[ f: \left[0, \frac{1}{2}\right] \rightarrow [0, 1] \quad \text{and} \quad g: \left[\frac{1}{2}, 1\right] \rightarrow [0, 1]. \]

Assume that the statement is proved for $n - 1$. The following two cases arise:

(1) \[ b = \frac{k + 1}{2^n} \equiv \frac{1}{2}, \quad \text{then} \quad f(a) = \frac{k}{2^n - 1}, \quad f(b) = \frac{k + 1}{2^n - 1} \equiv 1. \]

By our assumption there exists a polynomial function $p(x)$ such that $p\left(\frac{k}{2^n - 1}\right) = 0$, $p\left(\frac{k + 1}{2^n - 1}\right) = 1$, thus $p = pf$ satisfies $p(a) = 0$, $p(b) = 1$.

(2) \[ a = \frac{k}{2^n} \equiv \frac{1}{2}, \quad \text{then} \quad g(a) = \frac{k}{2^n - 1} - 1, \quad g(b) = \frac{k + 1}{2^n - 1} - 1 \]

and we have a polynomial function $p(x)$ such that $p\left(\frac{k}{2^n - 1} - 1\right) = 0$, $p\left(\frac{k + 1}{2^n - 1} - 1\right) = 1$.

Then $p = pg$ satisfies $p(a) = 0$ and $p(b) = 1$.

Now we consider the modular lattice $M_3[Q]$. The zero resp. unit of this lattice is denoted by $o$ resp. $i$. $M_3[Q]$ has three elements $u, v, w$ such that $o, u, v, w, i$ form a diamond, $M_3$.

We take $M_3[Q]$ in three pairwise disjoint copies $L_1$, $L_2$ and $L_3$. Then $o_k, u_k, v_k, w_k, i_k$ denote the elements corresponding to $o, u, v, w, i$ by the isomorphism $M_3[Q] \cong L^k$.

$J_1 = [u_1, l_1]$ is a principal dual ideal of $L_1$ and $J_1 \equiv Q$. Similarly, $J_0 = [o_2, w_2]$ is a principal ideal of $L_2$ isomorphic to $Q$. Therefore $J_1 \equiv J_2$, we can apply the Hall—Dilworth construction and we get the following modular lattice $L_{1,2}$:

\[ \text{Fig. 1} \]
Let $S$ be the sublattice of $L_{12}$ consisting of all elements $x \vee y$ where $x \equiv u_3$ and $y \equiv w_1$. The interval $[u_1, u_2]$ is isomorphic to $Q$ hence $S \equiv Q \times Q$. Similarly, $L_3$ contains a sublattice $T := \{x \vee y; x \equiv u_3, y \equiv w_3\}$ isomorphic to $Q \times Q$. Consequently we have an isomorphism $\varphi: S \to T$ with $\varphi(u_3) = u_3$, $\varphi(w_3) = w_3$. Now, we apply a gluing construction (similar to the Hall-Dilworth construction) by identifying the corresponding elements by $\varphi$. This construction was first defined in [4], see Fig. 2.

![Fig. 2](image)

In this way we get a modular lattice $L$.

![Fig. 3](image)

To prove that $L$ is modular, we have to show that $L$ does not contain a pentagon generated by $a, b, c$. But $L$ contains four sublattices generated by $\{u_3, v_1, x; 0 \leq x \leq v_3\}$, $\{u_3, w_1, x; 0 \leq x \leq v_3\}$, $\{v_3, w_1, x; 0 \leq x \leq v_3\}$ and $\{v_3, w_1, x; 0 \leq x \leq v_3\}$ which are all isomorphic to $M_5[Q]$, hence they are all modular sublattices. If $L$ contains a pentagon generated by $a, b, c$ then it is easy to see that $a, b, c$ are contained in one of these sublattices, a contradiction. The lattice $L$ contains the diamonds $(a_k, u_k, v_k, w_k, i_k)$, $k = 1, 2, 3$. We identify $[u_1, u_2]$ with $Q$, and define two polynomial functions on $L$:

$$f(x) = (((x \vee v_3) \wedge w_3) \vee v_3) \wedge u_2,$$

$$g(x) = (((x \vee v_3) \wedge w_3) \vee v_3) \wedge u_2.$$

It is easy to show that the restrictions of these functions to $Q$ are exactly the functions $f$ and $g$ defined above.

We prove that $L$ is locally order-polynomially complete. Let $a, b \in L$, $a < b$. By the gluing construction there exists a $c \in L$ such that $a \equiv c \equiv b$ and $a, c \in L_{1,2}$, $c, b \in L_3$ or conversely $a, c \in L_3$, $c, b \in L_{1,2}$. On the other hand $a < b$ implies that either $a < c$ or $c < b$. If $a, c \in L_{1,2}$, $a < c$ then either $a \wedge u_2 < c \wedge u_2$ or $a \wedge w_1 < c \wedge w_1$. Similarly if $c, b \in L_3$, $c < b$ then either $c \wedge u_2 < b \wedge u_2$ or $c \wedge w_1 < b \wedge w_1$. The intervals
[\omega_1, \omega_2] and [\omega_1, \omega_3] are projective and therefore we have a 1-place polynomial function \( t \) on \( L \) such that \( t(a) = t(b) \equiv u_2 \). (The second case, \( a, b \in L_3 \) is similar.) By Lemma 2 there exists a polynomial function \( s(x) \) satisfying \( u_2 = s(b) \), \( \omega_1 = s(a) \).

In \( L \) we have the polynomial function \( d(x) = (x \lor \omega_3) \land \omega_3 \) which satisfies \( d(u_2) = \omega_3 \), \( d(\omega_1) = \omega_1 \). Finally let \( r(x) = s(x) \lor d(x) \), then we have

\[
    r(a) = s(a) \lor d(a) = \omega_1 \lor \omega_1 = \omega_1,
\]

\[
    r(b) = s(b) \lor d(b) = \omega_2 \lor d(u_2) = \omega_2 \lor \omega_3 = \omega_2.
\]

We have to show that the conditions of Theorem B are satisfied. Indeed, let \( \Theta \) be a congruence relation of \( L \) such that \( a \equiv b(\Theta) \). Then we obtain \( \omega_1 = r(a) \equiv r(b) = \omega_2(\Theta) \), i.e. \( L \) is a simple lattice. If \( p, q \) are arbitrary 1-place polynomial functions then with the given polynomial function \( r(x) \) we get condition (2). The theorem is proved.

References


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MATHEMATICAL INSTITUTE  
OF THE HUNGARIAN ACADEMY OF SCIENCES  
BUDAPEST, REALTANODA U. 13—15.  
H-1053

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