ON A REPRESENTATION OF DISTRIBUTIVE LATTICES

E. T. SCHMIDT (Budapest)

§ 1. Introduction

The characterization problem of congruence lattices of lattices can be reduced to the representation of distributive join-semilattices as special join-homomorphic images of Boolean lattices. First, we formulate this problem. Let $D$ be an arbitrary finite distributive lattice and consider a finite Boolean lattice $B$ containing $D$. Then $D$ defines a closure operation $s : B \to B$ as follows:

$$s(x) = \land \{ y \in D ; y \geq x \}.$$ 

This closure operation has the additional property that for $x, y \in B$

$$s(x \lor y) = s(x) \lor s(y) \tag{1}$$

which means that $s$ is a topological closure operation. The sublattice $D$ is the set of all closed elements, i.e.,

$$D = s(B) = \{ s(x) ; x \in B \}.$$ 

Conversely, if $s$ is a topological closure operation on $B$ then the closed elements form a sublattice. We would like to represent all distributive (join-) semilattices on a "similar" way. If $s$ is a topological closure operation on an arbitrary Boolean lattice $B$ then it is easy to prove that $s(B)$ is a dual Heyting algebra, i.e., we cannot represent all distributive semilattices in this form. To overcome on this difficulties we follow a little modified way. (1) means that $s$ is a join-homomorphism from $B$ onto $D$. Let $h : B \to D$ be a join-homomorphism from the Boolean lattice $B$ into a distributive semilattice $D$. Then $h$ is called a distributive join-homomorphism if there is a family $\{ s_{i} ; i \in I \}$ of topological closure operations on $B$ such that the congruence kernel $\text{ker } h$ of $h$ is the join of the congruence kernels $\text{ker } s_{i}$, i.e.,

$$\text{ker } h = \lor \text{ker } s_{i}.$$ 

\textit{AMS (MOS) subject classifications} (1980). Primary 06D05; Secondary 06A12, 06E99.

\textit{Key words and phrases}. Distributive homomorphic images of Boolean lattices, distributive lattices, distributive semilattices.
In [2] I have proved the following:

**Theorem.** Every bounded distributive lattice is the distributive join-homomorphic image of a Boolean lattice.

In other words, if $D$ is a bounded distributive lattice, then there exist a Boolean lattice $B$ and a family $\{s_i; i \in I\}$ of topological closure operations on $B$ such that

$$D \cong B/\bigvee \ker s_i.$$  \hspace{1cm} (2)

The theorem is true if $D$ does not have a unit element, in this case $B$ denotes a generalized Boolean lattice. The related representation for distributive semilattices is still open. H. Dobbittin [1] has proved that every locally countable distributive semilattice can be represented in the form (2). The given representation has his own interest, but the most important consequence is the following: if $D$ can be represented in form (2) then the ideal lattice of $D$ is isomorphic to the congruence lattice of a lattice.

In this paper I shall give a new, relative short proof of this theorem. The most important part is the construction of a Boolean algebra $B$ which will be called the decomposition Boolean algebra of the given distributive lattice $D$.

**§ 2. Some properties of free Boolean algebras**

The decomposition Boolean algebra $B$ of $D$ is a special subalgebra of a free Boolean algebra which satisfies the property (2). First of all, we list some elementary, wellknown properties of the free Boolean algebras.

The free Boolean algebra $F(G)$ on a set $G$ of $n$ elements $g_1, \ldots, g_n$ is isomorphic to $2^n$ and each element may be expressed uniquely in disjunctive normal form, i.e., as a finite join of so called minimal terms:

$$\bigvee_{e} \{ g_{e_1}^1 \land \ldots \land g_{e_n}^n \},$$  \hspace{1cm} (3)

where $g_{e_i}^i$ is $g_i$ or $g_i'$ (the complement of $g_i$) and $e$ is a selection from the $2^n$ different distributions of dashes on the $g$'s. The minimal terms are the atoms of $F(G)$. I resp. 0 denote the unit resp. zero element. Let $G' = \{ g': g \in G \}$ where $G \cap G' = \emptyset$ and $g \rightarrow g'$ is a bijection between $G$ and $G'$. Let $(g')' = g'' = g$. For every natural number $k \leq n$ we define a subset of $F(G)$: $G_0 = = \{ 1 \}, G_1 = G \cup G'$ and

$$G_k = \{ x \in F(G), x = g_1^{e_1} \land \ldots \land g_n^{e_n} \}.$$
where \( g_1, \ldots, g_k \) are different elements of \( G \). For simplicity we can write \( x = g_1 \wedge \ldots \wedge g_k \), where \( g_1, \ldots, g_k \in G \) and assume that, for each \( i, j \), \( i \neq j \), \( g_i \neq g_j \). Further, let

\[
H = \bigcup_{i=0}^{n} G_i \cup \{0\}.
\]

Then \( H \) is closed under the meet operation of \( F(G) \). If we restrict the \( \vee \) operation to \( H \) on the usual way (i.e., for \( u, v, w \in H \) if \( u \vee v = w \) then we say that \( u \vee v \) is in \( H \) defined), we get a relative sublattice \( \langle H; \vee, \wedge \rangle \) of \( F(G) \).

It is easy to show that for incomparable \( u, v \in H \), \( u \vee v \) is defined if and only if there exist \( k \in \mathbb{N} \), \( w \in G_k \) and \( g \in G_1 \) such that \( u = w \wedge g \), \( v = w \wedge g' \). Then

\[
u \vee v = (w \wedge g) \wedge (w \wedge g') = w \wedge I = w.
\]

Obviously \( u, v \in G_{k-1} \). An ideal of a partial lattice is a nonvoid subset \( I \) such that

(i) if \( a, b \in I \) and \( a \vee b \) exists then \( a \vee b \in I \),

(ii) \( x \leq a \in I \) implies \( x \in I \).

It is easy to prove that \( F(G) \) is isomorphic to the lattice of all ideals of \( H \).

The description of the free Boolean algebra \( F(G) \) generated by an arbitrary (not necessarily finite) set \( G \) is similar, but in the infinite case there are no minimal terms (atoms) and therefore

\[
H = \bigcup_{i=0}^{\infty} G_i \cup \{0\}.
\]

In this case \( F(G) \) is the lattice of all finitely generated ideals of \( H \), and for every \( x \in F(G) \) there exists a smallest natural number \( n \) such that \( x \) has a uniquely representation in the from (3) with suitable \( g_1, \ldots, g_n \in G \). Obviously \( x \in F(\{g_1, \ldots, g_n\}) \).

We define a Boolean subalgebra of \( F(G) \) with a subset \( K \) of \( H \). Now, let us assume that \( K \) satisfies the following properties:

(4) \( 0 \in K \) and if \( x, y \in K \) then \( x \wedge y \in K \).

(5) If \( u \in K \cap G_{k+1} \), then there exists a \( v \in K \cap G_{k+1} \), \( v \neq u \) such that \( u \vee v \) is defined in \( H \) and \( u \vee v \in K \).

Let \( A \) be the set of all those elements of \( F(G) \) which have a representation as a finite join of elements of \( K \). Then \( A \) is obviously a sublattice of \( F(G) \). If \( u \in K \cap G_{k+1} \), then by (5) we have a \( v \in K \cap G_{k+1} \) such that \( u \vee v \) exists in \( H \) and \( u \vee v \in K \cap G_k \). Consequently, there exists a \( g \in G_1 \) satisfying \( u = (u \vee v) \wedge g \), \( v = (u \vee v) \wedge g' \) which involves

\[
u \wedge v = (u \vee v) \wedge g \wedge g' = 0.
\]
This means that \( v \) is the relative complement of \( u \) in the interval \([0, u \lor v]\). Similarly to \( u \lor v \in K \cap G_k \), there exists a relative complement \( z \in K \) in \([0, u \lor v \lor z]\) such that \( u \lor v \lor z \in K \cap G_{k-1} \). Then \( v \lor z \in A \) is the relative complement of \( u \) in \([0, u \lor v \lor z]\). After a finite number of steps this process breaks of by \([0, 1]\), i.e., every \( u \in K \) has a complement in \( A \) say \( u' = \lor k_i, k_i \in K \). Let \( v \) be another element of \( K \) then \( v' = \lor h_j \) with suitable \( h_j \in K \). Thus
\[
(u \lor v') = u' \land v' = \lor k_i \land \lor h_j = \lor (k_i \land h_j).
\]
By our assumption, \( K \) is closed under \( \land \), hence \( k_i \land h_j \in K \), i.e., \( u' \land v' \in A \). This proves that \( A \) is complemented, i.e., we have the following:

**Lemma 1.** Let \( K \) be a subset of \( H (\subseteq F(G)) \) which satisfies (4) and (5). Then the sublattice \( A \) generated by \( K \) is a \( \{0, 1\} \)-Boolean sublattice of \( F(G) \).

Any pair \((p, q), p \neq q\) of elements of a Boolean lattice \( B \) defines a closure operation \( C_{p,q} \) as follows:
\[
C_{p,q}(x) = \begin{cases} 
  x \lor q & \text{if } x \geq p \\
  x & \text{otherwise}.
\end{cases}
\]

It is easy to show that \( C_{p,q} \) is topological if and only if \( p \) is an atom. In a finite Boolean lattice a topological closure operation \( s(x) \) is determined by the closures of the atoms, consequently \( s(x) \) is the join of closure operations in the form \( C_{p,q}(x) \), where \( p \) and \( q \) are atoms. We need a special topological closure operation on \( F(G) \) which replace \( C_{p,q} \). Let \( p, q \in G_n \) for some \( n \) and let \( g \in G_1 \). Then
\[
p = (p \land g) \lor (p \land g'), q = (q \land g) \lor (q \land g')
\]
imply that
\[
C_{p,q} = C_{p\land g,q} \lor C_{p,q\land g'},
\]
hence
\[
C_{p,q}(x) \geq C_{p\land g,q}(x)
\]
for all \( x \) with \( x \neq C_{p,q}(x) \). This implies that the operation on \( F(G) \) defined by
\[
s_{p,q} = \lor \{ C_{p\land t,q\land t} \}_{t \in H}
\]
where \( p \land t, q \land t \neq 0 \) is a closure operation. (6) implies that \( s_{p,q} \) is topological.

**§ 3. The decomposition Boolean algebra**

Let \( D \) be a bounded distributive lattice. We construct from \( D \) a Boolean algebra \( B \), the decomposition Boolean algebra of \( D \). First of all we define the set \( G \) of generators of a free Boolean algebra \( F(G) \) and then we define a subset \( K \) of the corresponding relative sublattice \( H \subseteq F(G) \).
1 resp. 0 denote the unit resp. zero element of $D$. Let $G$ be a subset of $D \times D$ which contains no pairs $(a, a)$ and which contains to each two elements $a \neq b$ exactly one of the two pairs $(a, b)$ and $(b, a)$. If $g = (a, b) \in G$ then $g' = (b, a) \in G'$. Further, $G'$ and $H$ are defined as before, in § 2.

Let $h$ be a mapping from $G_1$ onto $D$ defined by

$$h((a, b)) = a \in D.$$ 

Every nonunit and nonzero element of $H$ has a unique representation as a meet of elements from $G_1$. Therefore we can extend $h$ to $H$ as follows: if $0 \neq x = g_1 \wedge \ldots \wedge g_k$ then

$$h(x) = h(g_1) \wedge \ldots \wedge h(g_k).$$

Further, let $h(1) = 1$ and $h(0) = 0$. In general, $h$ is not a join-homomorphism of the partial lattice $H$, e.g., if $(a, b) \in G_1$ and $a \lor b \neq 1$ in $D$ then $1 = (a, b) \lor \lor (b, a)$ in $H$ but

$$h(1) = 1 \neq a \lor b = h((a, b)) \lor h((b, a)).$$

We shall define the "greatest" relative-sublattice $K$ of $H$ such that the restriction of $h$ to $K$ will be a join-homomorphism.

The definition of $K$: First of all $0, 1 \in K$. Let $u \in G_k$, $k > 0$. Then $u \in K$ if and only if there exist $w \in K \cap G_{k-1}$ and $g = (a, b) \in G_1$ such that $u = w \wedge g$ and $h(u) \leq a \lor b$.

This definition implies that $g = (a, b) \in K$ iff $a \lor b = 1$.

First, we show that $K$ satisfies (4), i.e., it is closed under the $\wedge$-operation.

Let $T_k = K \cap \bigcup_{i=0}^k G_i$. We prove by induction on $k$ that $u_1, u_2 \in T_k$ implies $u_1 \wedge u_2 \in K$. If $u_1, u_2 \in T_i$ then we may assume that $u_1 \neq 1 \neq u_2$, i.e., $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2)$. Then $u_1, u_2 \in K$ means that $a_1 \lor b_1 = a_2 \lor b_2 = 1$. Consequently, $h(u_1) = a_1 \leq a_2 \lor b_2$, i.e., $u_1 \cup u_2 \in K$. Assume that our statement for $T_k$ is proved. If $u_1, u_2 \in T_{k+1}$ then $u_1 = w_1 \wedge g_1$, $u_2 = w_2 \wedge g_2$ where $g_i = (a_i, b_i)$ and $w_1, w_2$ are suitable elements of $T_k$ such that $h(w_i) \leq a_i \lor b_i$. By the assumption $w_1 \wedge w_2 \in K$. Then

$$h(w_1 \wedge w_2) \leq h(w_1) \leq a_1 \lor b_1$$

yields $w_1 \wedge w_2 \wedge g_1 \in K$. Similarly,

$$h(w_1 \wedge w_2 \wedge g_1 \wedge g_2) \leq h(w_2) \leq a_2 \lor b_2$$

implies

$$w_1 \wedge w_2 \wedge g_1 \wedge g_2 = u_1 \wedge u_2 \in K.$$
$K$ satisfies (5). Let $u \in K \cap G_{k+1}$. By the definition of $K$ there exist $w \in K \cap G_k$ and $g = (a, b) \in G_1$ such that $u = w \wedge g$ and $h(w) \leq a \vee b$. Let $v$ be the element $w \wedge g'$ where $g' = (b, a)$. Then again by the definition of $K$ we have $v \in K \cap G_{k+1}$ and $u \vee v = w$ is defined in $K$. This proves (5).

$K$ can be considered as a relative sublattice of $H$. The mapping $h: H \to D$ can be restricted to $K$, $h|_K: K \to D$. We prove:

Lemma 2. $h|_K: K \to D$ is a join-homomorphism onto $D$.

Proof. If $u$ and $v$ are incomparable elements of $K$ and $u \vee v$ is defined then

$u = g_1 \wedge \ldots \wedge g_{n-1} \wedge g$, $v = g_1 \wedge \ldots \wedge g_{n-1} \wedge g'$, $w = u \vee v = g_1 \wedge \ldots \wedge g_{n-1}$

where $g_1, \ldots, g, g' \in G_1$. We have to prove that $h(u \vee v) = h(u) \vee h(v)$. By the definition of $K$ there is a permutation of the elements $g_1, \ldots, g_{n-1}$, $g$, say

$g_1, g_2, \ldots, g_i, g, g_{i+1}, \ldots, g_{n-1}$

such that if $g = (a, b)$ then

$a \vee b \preceq h(g_1 \wedge \ldots \wedge g_i)$.

Consequently,

$a \vee b \preceq h(g_1 \wedge \ldots \wedge g_{n-1})$

and hence we conclude:

$h(u) \vee h(v) = (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge h(g)) \vee (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge h(g')) =

= (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge a) \vee (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge b) =

= h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge (a \vee b) = h(g_1 \wedge \ldots \wedge g_{n-1}) = h(u \vee v)$.

This proves our lemma.

By Lemma 2 we have a join-homomorphism $h|_K: K \to D$. $(a, 1) \in K$ and $h((a, 1)) = a$ which means that $h|_K$ is onto mapping. On the other hand the conditions of Lemma 1 are satisfied for $K$, thus the sublattice of $F(G)$ generated by $K$ is a Boolean lattice. We denote this by $B$. It is easy to see, that $h|_K: K \to D$ can be extend to a join-homomorphism $h: B \to D$, and $h$ is determined by its restriction to $K$, i.e., by $h|_K$.

We have to prove that $h$ is a distributive join-homomorphism. Assume that for $p \in K \cap G_K$ and $q \in K$, $p \neq q$ and $h(p) \geq h(q)$, i.e., $h(p \vee q) = h(q)$. Then in the free Boolean algebra $F(G)$

$s_{p,q}(p) = p \vee q (= C_{p,q}(p))$
and, for an arbitrary \( r \in K \cap G_K \), \( s_{pq}(r) = r \). Let us consider \( K \cap G_{K+1} \). If \( p \wedge q \in K \cap G_{K+1} \) for some \( g = (a, b) \) then, by the definition of \( K \), we have \( a \vee b \geq h(p) \). By our assumption, \( h(p) \geq h(q) \), thus \( a \vee b \geq h(q) \). This implies \( g \wedge g \in K \) and \( (p \wedge g) \vee (q \wedge g) \in B \). But

\[
C_{pqg}(p \wedge g) = (p \wedge g) \vee (q \wedge g),
\]

hence

\[
s_{pq}(p \wedge g) = C_{pqg}(p \wedge g) \in B.
\]

Thus \( s_{pq}(x) \in B \) for every \( x \in B \); i.e., \( s_{pq} \) is a topological closure operation on \( B \), and \( \ker h \supseteq \ker s_{pq} \). Consequently, \( \ker h = \vee \ker s_{pq} \), i.e., \( h \) is distributive.

REFERENCES


(Received January 13, 1985)