Polynomial automorphisms of lattices

E. TAMÁS SCHMIDT

Abstract. A polynomial automorphism of a lattice is a unary lattice polynomial \( f(x) \) for which the mapping \( x \rightarrow f(x) \) is an automorphism. It is proved that every bounded lattice with a finite automorphism group can be embedded as an ideal in a lattice \( K \) such that each automorphism of \( K \) is polynomial and there is a bijection between the automorphism groups of \( L \) and \( K \).

Keywords and phrases: Polynomial automorphism, automorphism group, lattice extension.
AMS-MOS subject classification: 06B15

1. Introduction. Ervin Fried and Harry Lakser [1] defined the concept of a polynomial automorphism of a lattice as a unary lattice polynomial \( f(x) \) for which the mapping \( x \rightarrow f(x) \) is an automorphism. They proved two theorems:

**Theorem A.** Each finite lattice \( L \) can be embedded in some finite lattice \( K \) such that the following three properties hold:

1. Each automorphism of \( K \) is polynomial;
2. Each automorphism of \( L \) extends to a unique automorphism of \( K \);
3. Each automorphism of \( K \) is the extension of an automorphism of \( L \).

For infinite lattices they could prove a weaker result:

**Theorem B.** Each lattice \( L \) can be embedded as a convex sublattice in some lattice \( K \) such that every automorphism of \( L \) is the restriction of a unary polynomial function on \( K \). If \( L \) has a \( 0 \) then this embedding is an ideal.

In this paper first we prove a little stronger version of Theorem A; the proof is slightly shorter than the original proof.

**Theorem 1.** Each finite lattice \( L \) can be embedded as a maximal filter (or maximal ideal) in some finite simple and atomistic lattice \( K \) such that the properties (1), (2) and (3) of Theorem A hold.

The main result of this paper is the generalization of Theorem A for arbitrary lattice having finite automorphism group:

**Theorem 2.** Let \( L \) be a bounded lattice with a finite automorphism group. \( L \) can be embedded as an ideal in some lattice \( K \) such that the properties (1), (2) and (3) of Theorem A hold.

If \( \alpha \in \text{Aut} \ L \) then a congruence relation \( \theta \) of \( L \) is called \( \alpha \)-admissible if \( x \equiv y(\theta) \) implies \( x^\alpha \equiv y^\alpha(\theta) \) \((x^\alpha \) denotes the image of \( x \) under \( \alpha \)). Similarly if \( G \) is a subgroup of \( \text{Aut} \ L \), then \( \theta \) is called \( G \)-admissible if \( \theta \) is \( \alpha \)-admissible for every \( \alpha \in G \).

* Research supported by Hungarian National Foundation for Scientific Research grant. no. 1813.
$G$-admissible congruence relations of $L$ from a $\{0, 1\}$-sublattice $\text{Con}_G(L)$ of $\text{Con}(L)$. Assume that $L$ is a convex sublattice of some lattice $K$ such that a given automorphism $\alpha$ of $L$ is the restriction of a unary polynomial function $f(x)$. A polynomial function is compatible, consequently if $\theta \in \text{Con}(L)$ can be extended to $K$ then $\theta$ is an $\alpha$-admissible congruence relation of $L$. We prove the following:

**Theorem 3.** Let $L$ be a lattice. $L$ can be embedded as a convex sublattice in some lattice $K$ such that the following properties hold:

1') Each automorphism of $L$ is the restriction of a unary polynomial function of $K$;

2') Each automorphism of $L$ extends to a unique automorphism of $K$;

3') Each automorphism of $K$ is the extension of an automorphism of $L$;

4') Each Aut $L$-admissible congruence relation of $L$ extends to a unique congruence relation of $K$;

5') Each proper congruence relation of $K$ is the extension of a congruence relation of $L$.

**2. Proof of Theorem 1.** Let $L$ be a finite lattice, with the zero element $0$. We add a new zero element $\bar{0}$ to $L$ (i.e. $\bar{0} < x$ for all $x \in L$), the resulting lattice is denoted by $\bar{L}$. Each automorphism of $L$ extends uniquely to an automorphism of $\bar{L}$ and each automorphism of $\bar{L}$ is the extension of an automorphism of $L$. For every $n \geq 1$ we consider the following lattice $S(n)$ (Fig. 1.)

![Diagram](attachment:fig_1.png)

**Fig. 1.**

It is easy to see that $S(n)$ is an atomistic simple lattice and has no nontrivial automorphism. We fix $n$ such that the length of $S(n)$ is greater than the length of $\bar{L}$ and we denote this $S(n)$ shortly by $S$. For every $u \in L$, $u \neq 0$, let $S_u$ be a lattice isomorphic to $S$, with the isomorphism $\varphi_u : S \rightarrow S_u$. Assume that $S_u \cap S_v = \emptyset$ if $u \neq v$ and $S_u \cap L = \emptyset$. We construct the extension $K$ of $L$ by gluing the lattices $S_u (u \in L)$ and $\bar{L}$: we identify the zero element of $S_u$ with $\bar{0} \in \bar{L}$ and the unit element of $S_u$ with $u \in \bar{L}$. Let $K$ be $\bar{L} \cup \{S_u; u \in L, u \neq 0\}$. The ordering in $S_u \subseteq K$ and $\bar{L} \subseteq K$ is the original, all these are sublattices of $K$. For $x \in S_u$, $x \not\in \bar{L}$ and $y \in \bar{L}$, $y \not\in S_u$, $x \leq y$ iff $u \leq y$ in $L$; if $u \not\leq y$ then $x$ and $y$ are incomparable and $\text{sup}\{x, y\} = u \lor y$ (the join in $L$) and $\text{inf}\{x, y\} = \bar{0}$. If $x \in S_u$, $y \in S_v$, $u \neq v$ and $x, y \in \bar{L}$ then $x, y$ are incomparable: $\text{sup}\{x, y\} = u \lor v$, $\text{inf}\{x, y\} = \bar{0}$. (See Fig. 2.)
L is the filter \([0]\), therefore \(L\) is a maximal filter of \(K\). Every element of \(S_u\) is the join of atoms, and these are atoms of \(K\), hence \(K\) is atomistic. Assume that \(\theta\) is a congruence relation of \(K\) and \(a = b(\theta), a \triangleright b\). We prove that \(\theta = \iota\) which means that \(K\) is simple. If \(a, b \in S_u\) for some \(u\) then the simplicity of \(S_u\) implies \(u = 0(\theta)\), therefore two different elements of \(\bar{L}\) are congruent. We may assume that \(a, b \in \bar{L}\). If \(a = 0\) let \(p\) be the element \(a\) (in this case \(b = 0\)). Otherwise let \(p\) be an atom of \(S_u\) and consider an arbitrary \(q \in S_1 \setminus \bar{L}\). Then \(a \equiv b(\theta)\) implies \(p = p \wedge a \equiv p \wedge b = 0(\theta)\), and \(1 = p \lor q \equiv 0 \lor q = q(\theta)\). \(S_1\) is simple, consequently \(1 = 0(\theta)\), i.e. \(\theta = \iota\).

By a theorem of Rudolf Wille [2] \(K\) is order–polynomially complete, i.e. every automorphism is polynomial which proves (1).

Let \(\alpha\) be an automorphism of \(L\) (i.e. of \(\bar{L}\)), and let \(x \in S_u, x \notin \bar{L}\). Then \(u = 0 \lor x, 0 = 0 \land x\). Assume that \(\alpha\) has an extension to \(K\), we denote one of this by the same letter. We conclude that \(u^\alpha = 0^\alpha \lor x^\alpha = 0 \lor x^\alpha\) and \(0 = 0^\alpha = 0 \land x^\alpha = 0 \land x^\alpha\) which imply \(x^\alpha \in S_u^\alpha\). Since \(S\) has no nontrivial automorphism \(x^\alpha\) is uniquely determined (i.e. \((\varphi_n(x))^\alpha = \varphi_n^\alpha(x))\). Conversely if we define \((\varphi_n(x))^\alpha = \varphi_n(x)\) for \(x \in S\) then we have an extension of \(\alpha\) to \(K\), which proves (2).

Finally let \(\beta\) be an automorphism of \(K\). 0 is an atom of \(K\), hence \(0^\beta\) must be an atom. By construction of \(K\) \(\text{length}(S) > \text{length}(L)\), which implies \(0^\beta \in \bar{L}\), i.e. \(0^\beta = 0\). If \(a \in L\) then \(a \geq 0\), thus \(a^\beta \geq 0^\beta = 0\), i.e. \(a^\beta \in L\). Consequently, the restriction of \(\beta\) to \(L\) is an automorphism \(L\) which proves (3).

3 Proof of Theorem 2. Let \(L\) be the given bounded lattice with the zero element \(u\) and unit element \(v\). Let \(\alpha\) be a fixed automorphism of \(L\). First we construct a lattice \(T_\alpha(L)\) such that \(L\) is an ideal of this lattice and \(L\) has a polynomial automorphism, which is an extension of \(\alpha\). We start with the following lattice \(T\), where the principal ideals \([a_i]\) and \([a_0]\) are isomorphic to \(S(2)\) resp. \(S(4)\) where \(S(n)\) denotes the lattice defined in the proof of Theorem 1. This lattice is a simple atomistic lattice.

It is easy to see that \(T\) has no nontrivial automorphism. We glue one–one copies of \(L\) into the prime intervals \([0, a_i], [a_i+1, b_i]\), \(i = 0, 2, 4\), i.e. we identify 0 with \(u\) and \(a_i\) with \(v\) (and similarly \(a_i+1\) with \(u, b_i\), with \(v\)). We fix some isomorphisms \(\varphi_i : L \rightarrow [0, a_i], \varphi_i : L \rightarrow [a_i+1, b_i]\) and we identify \(L\) with \([0, a_0]\), i.e. \(\varphi_0\) is the identity map. Let \(T_\alpha(L)\) be the set \(T \cup \bigcup_{i=0,2,4} ([0, a_i] \cup [a_i+1, b_i])\). We define a partial ordering on \(T_\alpha(L)\).
which will be an extension of the ordering of \( T \).

\[
\varphi_0(x) \leq \Psi_4(z) \quad \text{iff} \quad x \leq y^\alpha \quad \text{in} \quad L,
\]

and in the other cases:

\[
\begin{align*}
\varphi_i(x) &\leq \Psi_i(y) \\
\varphi_{i+2}(x) &\leq \Psi_i(y)
\end{align*}
\]

iff \( x \leq y \quad \text{in} \quad L \).

Then the subsets \([0, a_i] \cup [a_i, b_i], [0, a_{i+1}] \cup [a_{i+1}, b_l] \) are all isomorphic to \( L \times 2 \). It is easy to see that \( T_\alpha(L) \) is a lattice and has the following schematic diagram (Fig. 4.).

Let the unary polynomials \( f, g, h \) be defined by setting

\[
f(x) = ((\ldots (x \land a_0) \lor a_1) \land a_2) \lor a_3) \lor a_4 \land a_5,
\]
\[ g(x) = (\ldots (x \land a_2) \lor a_3) \land a_4) \lor a_5) \lor a_1) \land a_2, \]
\[ h(x) = (\ldots (x \land a_4) \lor a_5) \land a_0) \lor a_1) \lor a_3) \land a_4. \]

By the definition of \( T_\alpha(L) \) for an arbitrary \( x \in L \) (i.e. \( x \leq a_0 \)) \( f(x) = x^\alpha \), consequently \( \alpha \in \text{Aut} \ L \) is the restriction of the unary polynomial function \( f(x) \). We extend \( f(x) \) to a polynomial automorphism of \( T_\alpha(L) \). Define the unary polynomial \( f_\alpha(x) \) in \( T_\alpha(L) \) by setting:

\[ f_\alpha(x) = f(x) \lor (x \land a_2) \lor g(x) \lor (x \land a_3) \lor h(x) \lor (x \land a_5), \]

then \( f_\alpha(x) = f(x) \) if \( x \leq a_0 \) and similarly \( f_\alpha(x) = g(x) \) if \( x \leq a_2 \) and \( f_\alpha(x) = h(x) \) for \( x \leq a_4 \). The elements of \( T \subseteq T_\alpha(L) \) are all fixelements by \( f_\alpha(x) \).

Let \( \beta \) be an arbitrary automorphism of \( L \). We would like to extend \( \beta \) to \( T_\alpha(L) \), such that the elements of \( T \subseteq T_\alpha(L) \) remain fixed under the extension of \( \beta \). Then \( (f(x))^{\beta} = f(x^{\beta}) \), consequently \( x^{\alpha \beta} = x^{\beta \alpha} (x \in L) \). That means: \( \beta \) can be extended to \( T_\alpha(L) \) if \( \alpha \) and \( \beta \) commute. In the other case \( x^{\beta} \) must be a "new" element, therefore we define an extension \( K_\alpha \) of \( T_\alpha(L) \), such that every polynomial automorphism of \( T_\alpha(L) \) can be extend to a polynomial automorphism of \( K_\alpha \).

Let \( C \) be the centralizer of \( \alpha \) in \( G = \text{Aut} \ L \). \( C = C_0, C_1, \ldots, C_n \) denote the right cosets of \( C \). For every \( C_i \) we consider an isomorphic copy \( M_i \) of \( T_\alpha(L) \) and we fix for every \( i \) an isomorphism \( r_i : T_\alpha(L) \rightarrow M_i \). We identify \( M_0 \) with \( T_\alpha(L) \). \( L \) is an ideal of \( T_\alpha(L) = M_0 \). Let \( L_i \) be the image of \( L \) by \( r_i \), then \( L_i \) is an ideal of \( M_i \). We glue together the lattices \( M_i (i = 0, 1, \ldots, n) \) by identifying the ideals \( L_i \). Then \( M_i \cap M_j = L \) if \( i \neq j \). Let \( 1_i \) be the unit element of \( M_i \) and let \( 1 \) be the unit of \( L \), then \( 1_i \land 1_j = 1 \). Finally, we adjoin a new unit element \( I \) to the poset \( \cup M_i \). Let \( K_\alpha \) be the poset \( \{ I \} \cup \cup M_i \). This \( K_\alpha \) is obviously a lattice, every \( M_i \) is an ideal of \( K_\alpha \) and if \( x \in M_i, y \in M_j, x, y \not\in L \) then \( x \lor y = I \). (see Fig. 5).

\[ \text{Fig. 5.} \]

We prove that \( K_\alpha \) satisfies the following three properties:

(i) \( \alpha \) is the restriction of a polynomial automorphism of \( K_\alpha \);

(ii) Each (polynomial) automorphism of \( L \) extends to a unique (polynomial) automorphism of \( K_\alpha \);

(iii) Each automorphism of \( K_\alpha \) is the extension of an automorphism of \( L \).
Using these properties the proof of the theorem is very easy. By our assumption \( G = \text{Aut} L \) is finite. Consider an \( \alpha \in G \), then we have extension \( K_\alpha \) of \( L \). \( \beta \in G \) can be extended to \( K_\alpha \), i.e. \( \beta \) is an automorphism of \( K_\alpha \). We apply the same construction starting with \( K_\alpha \) and we get the extension \( (K_\alpha)_\beta \). By induction we have finally an extension \( K \) of \( L \) which satisfies (1)--(3). \( L \) is an ideal of \( K_\alpha \), hence \( L \) is an ideal of \( K \).

To prove (i) let \( f_\alpha \) be the polynomial automorphism of \( M_1 \) which corresponds to \( f_\alpha = f_\alpha^0 \) by the isomorphism \( \tau_1 : T_\alpha(L) \to M_1 \). Define \( F_\alpha(x) = \bigvee_{i=0}^n f_\alpha^i(x \wedge 1) \) then the restriction of \( F_\alpha(x) \) to \( M_1 \subseteq K_\alpha \) is \( f_\alpha^i \) consequently its restriction to \( L \subseteq M_0 \) is \( \alpha \) which proves (i).

We prove (ii). Every element \( y \notin \{1, \tau, s, t\} \) of \( T_\alpha(L) = M_0 \) has a unique representation in the following form:

\[
y = ((\ldots (x \wedge a_0) \lor a_1) \ldots) \lor a_k \quad \text{where} \quad x \in L, k \leq 5.
\]

If \( \beta \) has an extension to \( K_\alpha \) — we use same notation — then

\[
y^\beta = ((\ldots (x^\beta \wedge a_0^\beta) \lor a_1^\beta) \ldots) \lor a_k^\beta,
\]

which means that \( y^\beta \in C_i \) where \( \beta \in C_i \), hence \( y^\beta = \tau_i(y) \). Let \( \gamma \) be an arbitrary automorphism of \( L \), and let \( \beta \gamma \in C_j \). Then let \( (y^\beta)^\gamma = \tau_j(y) \), which is an automorphism of \( K_\alpha \). This proves that \( \beta \) has a unique extension to \( K_\alpha \). (I is by the extension obviously a fixelement.)

Assume that \( \beta \) is a polynomial automorphism of \( L \). We prove that the extension of \( \beta \) — which will be denoted again by \( \beta \) — is a polynomial automorphism of \( K_\alpha \). \( T \) is a sublattice of \( T_\alpha(L) = M_0 \), i.e. we have the embedding \( \epsilon : T \to M_0 \). Applying the isomorphism \( \tau_1 : M_0 \to M_1 \) we get the sublattices \( T_i = \tau_1(T) \) of \( M_1 \). Obviously \( T_i \cap T_j = \{0, a_0\} \) if \( i \neq j \). Let \( T_{\alpha}^* \) be the sublattice \( \bigcup T_i \cup \{\} \) of \( K_\alpha \). This is a simple atomistic lattice, consequently by [2] every automorphism of \( T_{\alpha}^* \) is polynomial.

Consider the elements \( a_i(i = 0, 1, \ldots, 5) \) of \( T \) (see Fig. 3.). Let \( \gamma \) be an arbitrary automorphism of \( K_\alpha \), then \( a_i^\gamma \in T_k \subseteq M_k \) for some \( k \). If we apply the automorphism \( \beta \) then \( a_i^\gamma \beta \in T_{l} \subseteq M_l \) for some \( l \). We discuss two different cases. First assume, that \( i = 0, 2 \) or 4. The intervals \([0, a_0]\) and \([0, a_1]\) are projective in \( T_\alpha(L) \), consequently \([0, a_1^\gamma]\) and \([0, a_1^{\gamma\beta}]\) are projective in \( K_\alpha \). That means, we have a unary polynomial which transposes \([0, a_0]\) onto \([0, a_1^{\gamma\beta}]\). Now, let \( i = 1, 3 \) or 5. Then the principal ideals \( [a_i]\) belong to \( T \) i.e. \([0, a_i]\), \([0, a_i^{\gamma\beta}]\) are contained in \( T_{\alpha}^* \). But every automorphism of \( T_{\alpha}^* \) is polynomial, i.e. we have again a unary polynomial which transposes \([0, a_i]\) onto \([0, a_i^{\gamma\beta}]\). This proves that \( \beta \) is a polynomial automorphism of \( K_\alpha \).

Finally we prove (iii). Let \( \gamma \) be an arbitrary automorphism of \( K_\alpha \). The unit element, \( 1 \in L \) must be a fixelement of \( \gamma \) (1 is the intersection of dual atoms). Consequently the restriction of \( \gamma \) to \([1] = L \) is an automorphism.

4. Proof of Theorem 3. The proof is similar to the proof of Theorem 2, but we start with an other lattice \( T \):

The principal ideals \( a_2 \) and \( a_4 \) are isomorphic to \( S(2) \) resp \( S(4) \) where \( S(n) \) is the lattice defined in the proof of Theorem 1. \( T \) has only the trivial automorphism and
has exactly one non trivial congruence relation with the congruence classes \( \{a_i, b_i\} \), \( i = 0, \ldots, 5 \). Let \( L \) be the given bounded lattice, with zero element \( u \) and unit element \( v \). We glue one–one copies of \( L \) into the prime intervals \([a_i, b_i]\) \( (i = 0, \ldots, 5) \) identifying \( u \) with \( a_i \) and \( v \) with \( b_i \). We fix the isomorphisms \( \varphi_i : L \to [a_i, b_i] \), \( \varphi_0 \) is the identify map i.e. \( L = [a_0, b_0] \). Let \( \alpha \) be an automorphism of \( L \). The ordering relation is defined as follows

\[
\begin{align*}
\varphi_2(x) &\leq \varphi_{i+1}(y) \quad \text{iff } x \leq y \quad \text{in } L \\
\varphi_1(x) &\geq \varphi_{i+1}(y) \quad \text{iff } y \leq x \quad \text{in } L \\
\varphi_0(x) &\leq \varphi_2(y) \quad \text{iff } x \leq y^\alpha \quad \text{in } L
\end{align*}
\]

We denote this poset by \( T_\alpha(L) \). It is easy to see that \( T_\alpha(L) \) is a lattice. If

\[ f(x) = ((\ldots (x \land b_0) \lor a_1) \land b_2) \lor a_3 \cdots \lor a_5 \land b_0 \]

then its restriction to \( L \) is \( \alpha \). This \( f \) is obviously not a polynomial automorphism of \( T_\alpha(L) \).

As in the construction given in the proof of Theorem 2 we extend \( T_\alpha(L) \) to a lattice \( K_\alpha \). Let \( C \) be the centralizer of \( \alpha \) in \( G = Aut L \), and let denote \( C = C_0, \ldots, C_n \) the right cosets. For every \( i \) we consider an isomorphic copy \( M_i \) of \( T_\alpha(L) \) with a fixed isomorphism \( r_i : T_\alpha(L) \to M_i \). Finally we identify \( M_0 \) with \( T_\alpha(L) \) i.e. \( r_0 \) is the identify map. \( L \) is a convex subset of \( M_0 \). The image of \( L \) by \( r_i \) is a convex sublattice \( L_i \) of \( M_i \). We glue together the lattices \( M_i (i = 0, \ldots, n) \) by identifying the sublattices \( L_i \), then \( M_i \cap M_j = L \) if \( i \neq j \). Finally we adjoin a new unit \( I \) and a new zero \( 0 \) to the poset \( \bigcup M_i \). Let \( K_\alpha \) be the poset \( \{ I, 0 \} \cup \bigcup M_i \). It is easy to see that \( K_\alpha \) is a lattice and every \( M_i \) is a convex sublattice of \( K_\alpha \). Similar to the proof of Theorem 2 we can show:

(i') \( \alpha \) is the restriction of a unary polynomial function;

(ii') Each automorphism of \( L \) extends to an automorphism of \( K_\alpha \).

(iii') Each automorphism of \( K_\alpha \) is the extension of an automorphism of \( L \).

In \( K_\alpha \) every non unit congruence relation is determined by its restriction to \( L = [a_0, b_0] \) which proves (4') and (5') in the theorem.
REFERENCES