A NEW LOOK AT THE SEMIMODULAR LATTICES
A GEOMETRIC APPROACH
(NOT FINISHED MANUSCRIPT)

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ABSTRACT. This paper is an experiment, how it is possible to treat semimodular lattices as geometric shapes. In 2009 I published a paper with Gábor Czédli, [4], we proved that every semimodular lattice $L$ can be obtained from a direct power of a chain $G = C^n$ – geometrically a cube – on an easy way. $L$ is the cover-preserving join-homomorphism of $G$.

We introduce the concept of rectangular semimodular lattices and prove that the building stones of semimodular lattices are special rectangles, the pigeonholes. The building tool is a special S-verklebte sum (introduced by Christian Herrmann [21]) this is the patchwork.

This manuscript includes parts of the papers [2], [3], [4], [5], [6], [7], [8], [9], [10] and [11].

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PART I

Cover-preserving join-congruences
Part 1. Cover-preserving join-congruences

1. Preliminaries

This paper is a geometric approach to finite semimodular lattices. We consider the diagrams of semimodular lattices as geometric shapes. In this approach the eight element boolean lattice is a cube.

Let $L$ and $K$ be finite lattices. A join-homomorphism $\varphi : L \to K$ is said to be cover-preserving iff it preserves the relation $\preceq$. Similarly, a join-congruence $\Phi$ of $L$ is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]_{\Phi}$ is cover-preserving. $J(L)$ denotes the order of all nonzero join-irreducible elements of $L$ and $J_0(L)$ is $J(L) \cup \{0\}$.

The concept of the dimension of a semimodular lattice is a sensitive step in this paper. There are different possibilities. The width $w(P)$ of a (finite) order $P$ is defined to be $\max\{n: P$ has an $n$-element antichain$\}$. If $L$ is a lattice the number $k = w(J(L))$ is called the $L$-width of $L$. This will be denoted by $\dim(L)$. In virtue of Dilworth [12], $P$ is the union of $k$ appropriate chains. This concept is a kind of geometric dimension. The Kurosh-Ore dimension $\Dim(L)$ of $L$ is the minimal number of join-irreducibles needed to span the unit element of $L$.

The dimension of a finite semimodular lattice $L$ is the greatest natural number $n$ such that there is an interval $I$ with the Kurosh-Ore dimension $n = \Dim(I)$.

It was proved by G. Czédli and E. T. Schmidt [4], (for planar semimodular lattices see G. Grätzer, E. Knapp [15], [16],[17],[18]):

**Theorem 1.** Each finite semimodular lattice $L$ is a cover-preserving join-homomorphic image of a distributive lattice $G$ which is the direct product of $\dim(L)$ finite chains.

**Remark.** The theorem was proved originally (and independently) in the first edition of M. Stern, [28] (Th6.3.14, p. 240).

$G$ can be interpreted in two different ways:

1. $G$ is a grid (i.e. a coordinate system),
2. $G$ is a geometric rectangle (other names: a $n$-dimensional cube, a shape or cuboid).

By Theorem 1 we can get every semimodular lattice $L$ from a rectangle, using as tool a cover-preserving join homomorphism. We ”carve” the semimodular lattice $L$ from $G$.

In [4] we proved an other similar theorem:

**Theorem 2.** Every finite semimodular lattice $L$ is a cover-preserving join-homomorphic image of the unique distributive lattice $D$ determined by $J(D) \cong J(L)$. Moreover, the restriction of an appropriate cover-preserving join-homomorphism from $D$ onto $L$ is a $J(D) \to J(L)$ order isomorphism.

From the proof of Theorem 2 we can see that $F = H(J(L))$, which is the distributive lattice determined by the order $P = J(L)$ and the poset $H(P)$ denotes the lattice of all downsets of the order $P$.

A subset $\{a, b, a \land b, a \lor b\}$ of a semimodular lattice is called a covering square if $a \land b \prec a$ and $a \land b \prec b$. The semimodularity implies $a \prec a \lor b$ and $b \prec a \lor b$. 
If we have a grid \( G = C_k^n \) (\( C_k = \{1, 2, \ldots, k\} \) is the chain of natural numbers), then we will denote its elements as vectors \( q = (x_1, x_2, \ldots, x_n) \).

We need the following lemma [4], which characterize the cover-preserving join-homomorphisms:

**Lemma 1. (Covering square lemma.)** Let \( \Phi \) be a join-congruence of a finite semimodular lattice \( M \). Then \( \Phi \) is cover-preserving if and only if for any covering square \( S = \{a \wedge b, a, b, a \vee b\} \) if \( a \wedge b \not\equiv a \) (\( \Phi \)) and \( a \wedge b \not\equiv b \) (\( \Phi \)) then
\[
a \equiv a \vee b \text{ (} \Phi \text{)} \text{ implies } b \equiv a \vee b \text{ (} \Phi \text{)}.
\]

This lemma states: a join-congruence of a finite semimodular lattice \( M \) is cover-preserving if and only if its restriction to any covering square is a cover-preserving join-congruence.

If \( (L) \) denotes the length of \( L \). Let \( a/b \) and \( c/d \) prime quotients of a lattice \( L \). If \( b \vee c = a, b \wedge c = d \) we say that \( a/b \) is perspective up to \( c/d \) and we write \( a/b \searrow c/d \).

### 2. The heuristic approach

#### 2.1. Distributive lattices.

First, we consider the simplest case. A planar distributive lattice \( D \) is the cover-preserving sublattice of a direct product of two finite chains, which is geometrically a rectangle. On Figure 1 you can see a typical example (\( D \) is the shaded part), this lattice contains three maximal rectangles \( A = [c, 1], B = [a, d] \) and \( C = [0, b] \) which are glued together. The intersection (which is a rectangle) \( A \cap B \) is a dual ideal of \( A1 \) and an ideal of \( B \). *Every planar distributive lattice \( D \) can be covered by rectangles, that is, in other words it is a ”glued system” of rectangles.* This gluing is a special case of the S-verklebte sum, introduced by Christian Herrmann [21] for modular lattices. We can do this in various ways such as with largest rectangular components (on Figure 1 there are three components).

If we cut up the lattice defined on Figure 1 into smaller pieces, we may assume that the intersection of two ”small” rectangles is either empty or is a part of a side of the rectangular components. The smallest rectangles are obviously the unit squares. This means a planar distributive lattice \( D \) looks like the mosaic pavement in the kitchen, the only one difference is that on some places we may have ”degenerate” unit squares (blocks) which are unit sections or one element (see Figure 2). On Figure 1. 17 blocks \( M_1, \ldots, M_{17} \). This covering has the following special property: if \( M_i \cap M_j \neq \emptyset, i \neq j \), then the union, \( M_i \cup M_j \) is the Hall-Dilworth gluing via an edge or corner, i.e. the ”dimension” of \( M_i \cap M_j \) is smaller then the ”dimension” of \( M_i \) and \( M_j \). In \( G = 3 \times 3 \) on Figure 2 we have four blocks \( M_1, \ldots, M_4 \). If we take the cover-preserving join-homomorphic image, \( S_7 \), then we have again four mosaics \( M_1, \ldots, M_4 \), but \( M_4 \) is in this case a degenerate mosaic, the edge \( \overline{1} \). A 1-narrows of \( L \) is an element \( a \in L - \{0, 1\} \) such that \( L = \downarrow a \cup \uparrow a \). A 2-narrows of \( L \) is a priminterval \( [u, v] \) such that \( u \) is meet-irreducible and \( v \) is join-irreducible. If the planar distributive lattice \( D \) has no \( n \)-narrows, \( n = 1, 2 \) we say that \( D \) is **narrows free**.

Every finite distributive lattice \( L \) (not only the planar) can be considered as a shape which we obtain from cover-preserving cubes (bricks) applying gluing by faces. We will work with such coverings by semimodular lattices and we will call them **patchwork**. The patchwork was introduced for planar semimodular lattices in [10].
We will define special semimodular lattices the *pigeonholes*. In the class of finite distributive lattices these are the Boolean lattices. Using these terminologies we can say:

*Every finite distributive lattice is the patchwork of pigeonholes.*

Our goal to extend these trivial statement for all semimodular lattices. As a non-distributive example let us take the lattice (patchwork [10]) on Figure 3.

Let us take again Figure 1. If we start with $D$ (the shaded part) then we glue to this further rectangulars to $E$, $F$ and $G$ to get a "narrowest big" rectangle. This is a rectangle hull of $D$. We extend these trivial statement for all semimodular lattices.

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**Figure 1.** A "tipical" planar distributive lattice $D$ (the shadded part)

2.2. **The breakdown procedure.** There is another way to represent a finite distributive lattice $L$, this is the *breakdown procedure*: we cut out (carve) from the "big" cuboid $E = \{0, c_5, c_3, 1\}$ (we call *envelope* distributive lattice is the cover-preserving sublattice of a rectangle (cuboid) $R$ such that $Jw(R) = Jw(L)$. In the planar case, see Figure 1, we cut out $S_1$ and $S_2$. It is easy to define the *corner* and other geometric concepts for arbitrary finite distributive lattice which was defined originally in planar case in [17].

Our goal is to give "similar" representations for all semimodular lattices. We ask, which properties of special gluings are inherited for semimodular lattices?
First, in section 3 we define distributive lattices which are associated to a semimodular lattice.

2.3. Minimal cover-preserving join-congruences. Let \( s, t \) be elements of a semimodular lattice \( L \). Then \( \operatorname{con}_{\text{cp}}(s, t) \) denotes the smallest cover-preserving join-congruence \( \alpha \) of \( L \) where \( s \equiv t \) (\( \alpha \)). Let \( t \prec s \) be a covering pair. It is clear that for a covering pair \( d \prec c, c \equiv d \) (\( \alpha \)) if and only if there is an lower cover \( u \) of \( s \) such that \( c/d \) is perspective up to \( s/u \). That means \( \alpha \) is determined by the element
s and we write:

\[ \con^{\text{cp}}(s) := \con^{\text{cp}}(s, t) \]

On Figure 7 you can see the planar case. Obviously,

\[ \con^{\text{cp}}(s) = \bigvee_{t_i \prec s} (\con^\vee(s, t_i)). \]

where \( \con^\vee(s, t) \) denotes the smallest join-congruence \( \alpha \) of \( L \) where \( s \equiv t \) (\( \alpha \)).

If we have three chains the \( \con^{\text{cp}}(s) \) is given on Figure 8.

3. **Distributive lattices associated to a semimodular lattice**

Let \( S \) be a semimodular lattice. By Theorem 1 and Theorem 2 there are different distributive lattices such that \( S \) is the cover-preserving join-homomorphic image of
these distributive lattices. How can we get these lattices and what is the connection between these lattices?

3.1. The grids and the frame. Let $S$ be a finite semimodular lattice and let $C_1, \ldots, C_n$ be maximal chains of $S$ such that $n = \dim(S)$ and $C_1 \cup C_2 \cup \ldots \cup C_n \supseteq \mathbf{J}(S)$.

Definition 1. The direct product $G = C_1 \times \ldots \times C_n$ is called a grid of $S$. 
The Jordan-Hölder theorem implies that the grid is determined up to isomorphism. Theorem 1 asserts that \( S \) is a cover-preserving join-homomorphic image of \( G \).

Let \( D_1, \ldots, D_n \) be pairwise disjoint subchains of \( \text{J}(S) \) such that \( n = \text{w}(\text{J}(S)) \) and \( D_1 \cup D_2 \cup \ldots \cup D_n = \text{J}(S) \). We may assume that \( D_i \subset C_i \), i.e. the \( D_i \) is the restriction of \( C_i \) to \( \text{J}(S) \).

**Definition 2.** The direct product \( G_l = D_1 \times \ldots \times D_n \) is called a lower grid of \( S \).

\( G_l \) is a sublattice of \( G \). Let us note that the lower grid is not determined uniquely. Obviously, geometrically the grid is a cuboid (rectangular). More results on grids see in G. Czédli, [3]. If \( G \) is fixed then \( G_l \) is determined.

It is clear that \( \text{l}(G) = n \cdot \text{l}(S) \) and \( \text{l}(G_l) = |\text{J}(S)| \). The last equality means that every finite distributive lattice \( D \) is the cover-preserving sublattice of the direct product of \( \text{dim}(D) \) chains, this is the lower grid. The lower grid of \( M_3 \) is the \( 2^3 \) boolean lattice and the grid is isomorphic to \( 3^3 \).

**Definition 3.** The frame of the semimodular lattice \( S \) is \( H(\text{J}(S)) \).

It is clear that \( \text{J}(H(\text{J}(S))) \) is order isomorphic to \( \text{J}(S) \). The frame is a cover-preserving sublattice of the lower grid, \( \text{Frame}(S) \subseteq G_l \subseteq G \). It is easy to see that \( \text{w}(\text{Frame}(S)) = \text{w}(S) \).

### 3.2. The skeleton.

The skeleton was introduced by G. Grätzer and R. W. Quackenbush [20] for planar modular lattices (they called them frame). Let \( S \) be a planar modular lattice with a planar diagram \( P \), and let \( M = [b, t] \) be an interval isomorphic to \( M_n \) with exterior atoms \( \{a_1, a_2\} \) and interior atoms \( \{a_2, \ldots, a_{n-1}\} \) between \( a_1 \) and \( a_2 \) in the planar diagram \( P \). These are doubly irreducible elements. We say that \( a \in S \) is an internal element (with respect to \( P \)) if \( a \) is an external element. Let \( \text{Skeleton}(S) \) be the sublattice of \( S \) consisting of all external elements of \( S \), which is unique determined up to isomorphism.

We extend this concept for semimodular lattices.

**Definition 4.** A maximal cover-preserving complemented sublattice \( C \) in a distributive lattice \( D \) is called a cell. If \( C \cong 2^k \) we say that \( C \) is a \( k \)-cell.
Definition 5. The skeleton, Skeleton(S) of a finite semimodular lattice S is a maximal cover-preserving distributive sublattice D of S such that every element of S is in the interval generated by a cell of D.

Let us show some examples. We consider the skeleton as a box and the semimodular lattice is packed in this box see Figures 5, 9, 10, 15, 20, 21, 22, 38, 40, 43. (Let us remark that not every semimodular lattice has a skeleton.)

**Figure 9.** The skeleton of a semimodular lattice S

For other examples see Figure 20 and the Fano space is presented on Figure 39.

Definition 6. A finite distributive lattice D is called connected if to arbitrary two cells C_1 and C_n there is a sequence C_2, ..., C_{n-1} such that |C_i \cap C_{i+1}| > 1 for every i.

The following theorem generalize a results of [20]:

**Theorem 3.** Every connected finite distributive lattice D is isomorphic to the skeleton of a simple modular lattice.

Proof. Similar to [20]. □

3.3. Mappings associated to grids and frames. con^∨(y, x) denotes the principal join-congruence, i.e. the smallest join-congruence – under which y ≡ x.

Definition 7. A join-congruence Φ of a distributive lattice D is called distributive join-congruence if Φ = ∨ con^∨(p_i, x_i), where p_i ∈ D are join-irreducible elements.

This notion was introduced in [23].

Let S be a semimodular lattice, then we have the distributive lattices G, G_l, and the Frame(S) = H(J(S)). There are several mappings between these lattices:

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & S \\
\downarrow{\alpha} & & \uparrow{\beta} \\
G_l & \xrightarrow{\psi} & \text{Frame}(S)
\end{array}
\]

(I.) \( \varphi \) from G onto S. Theorem 1 states that there is cover-preserving join-homomorphism \( \varphi \) from G onto S:

\[ \varphi : G \to S, \quad (x_1, \ldots, x_k) \mapsto x_1 \lor \cdots \lor x_k. \]
(II.) \( \alpha : G \rightarrow G_l \). We may assume that \( D_i \subseteq C_i \) for every \( i \in \{1, \ldots, n\} \). In this case to every \( c \in C_i \) there is a largest \( c^* \in D_i \) such that \( c \geq c^* \).

It is clear that \( \alpha \) is a lattice homomorphism of \( G \) into \( G_l \) (which is determined by the congruence \( (c_1, \ldots, c_n) \equiv (c_1^*, \ldots, c_n^*)(\Theta) \)).

(III.) \( \psi : G_l \rightarrow \text{H}(J(S)) = \text{Frame}(S) \). This is a distributive if join-congruence.

Take for simplicity the \( n = 2 \) case. By the definition of \( G_l \) every \( p \in J(S) \) appears in one of the \( D_i \)-s. If \( p, q \in D_i, p < q \), then \( p < q \) in \( J(S) \) too. If \( p \in D_i \) and \( q \in D_j, i \neq j \) but \( p < q \) in \( J(S) \) then take the elements \( (p, 0), (p, q) \) of \( G_l \) and the join-congruence induced by this pair

\[
\text{con}^\vee((p, 0), (p, q))
\]

On this way we get a join-homomorphism from \( G_l \) to \( \text{Frame}(S) \), i.e. the frame \( \text{Frame}(S) \) is distributive join-homomorphic image of the lower grid.

(IV.) \( \beta : \text{Frame}(S) = \text{H}(J(S)) \rightarrow S \) exists by Theorem 2, \( \beta \) is a cover-preserving join-homomorphism.

**Lemma 2.** \( \varphi = \beta \psi \alpha \).

**Proof.** Easy computation. \( \square \)

### 3.4. Horizontal and vertical edges of an order.

Let us consider \( P = J(D) \), where \( D \) is the distributive lattice defined on Figure 1. In the diagram (graph) of \( P \) (see Figure 10) we have two type of edges: \( x, y, x < y, x, y \in D_i \), for some \( i \), these are the vertical edges (these are in the two chains) and all others the horizontal edges. We omit (cut out) the horizontal edges of \( J(D) \) we get the diagram of \( J(G_l) \), i.e. we have two maximal chains \( 0 < c_1 < c_2 < c_3 < c_4 < c_5 \) and \( 0 < e_1 < e_2 < e_3 \) and the relations, \( e_2 > c_3, c_4 > e_1 \) which are given by two horizontal edges, \( e_2, c_3 \) and \( e_4, e_1 \) (red lines).

![Figure 10. J(D)](image)

On \( G_l \) take the corresponding join congruence

\[
\Theta = \text{con}^\vee((c_4, 0), (c_4, e_1)) \vee \text{con}^\vee((0, e_2), (c_2, e_2))
\]

Then \( G_l/\Theta = P \). The natural join-homomorphism \( G_l \rightarrow G_l/\Phi \), preserves boundary chains, this observation makes possible to introduce such notions in an easy way
in the \(J\)-width > 2 cases. To the horizontal edge \(e_3, e_1\) we associate the cuboid \(S_1\) and similarly, \(e_4, e_1\) represents \(S_2\). We have a bijection between the horizontal edges and "corner" cuboids of \(G_1\).

We are going to the \(w(P) = 3\) case and take the following poset \(P\) see on Figure 11:

![Figure 11. Two horizontal edges](image)

We can visualize this as follows, see Figure 12. From the "big" cuboid we cut out two "small" cuboid-s, which are determined by the horizontal edges.

![Figure 12. Removing corners from a cuboid](image)

It is easy to prove that every finite distributive lattice \(D\) of \(J\)-width = \(n\) is

1. a cover-preserving sublattice of a rectangular lattice \(R\) of \(\text{dim}(D) = n\) (envelop),
2. the \(S\)-glued sum of rectangular components of \(J\)-width \(k \leq n\).

It is easy to define the volume and surface area of an arbitrary finite distributive lattice.

Is something like true for all semimodular lattices?
3.5. **Rectangular lattices.** Rectangular lattices were introduced by G. Grätzer and E. Knapp [15] for planar semimodular lattices: a left corner (resp. right corner) of a planar lattice $K$ in a double-irreducible element in $K - \{0, 1\}$ on the left (resp., right) boundary of $K$. A rectangular lattice $L$ is a planar semimodular lattice which has exactly one left corner, $u_l$ and exactly one right corner, $u_r$ and they are complementary – that is, $u_l \lor u_r = 1$ and $u_l \land u_r = 0$. The direct product of two chains is rectangular.

We introduce this notion for arbitrary finite semimodular lattice. Let $X, Y$ be posets. The cardinal sum $X + Y$ of $X$ and $Y$ is the set of all elements in $X$ and $Y$ considered as disjoint. The relation $\leq$ keeps its meaning in $X$ and in $Y$, while neither $x \geq y$ nor $x \leq y$ for all $x \in X, y \in Y$.

**Definition 8.** A rectangular lattice $L$ is a finite semimodular lattice in which $J(L)$ is the cardinal sum of chains.

Geometric lattices are rectangular. In [7] we introduced the almost geometric lattices these are lattices in which $J(L)$ is the cardinal sum of at most two element chains. In the class of finite distributive lattices the rectangular lattices are the the direct products of chains. The lattices $M_3[C_n]$ are modular rectangular lattices ($C_n$ is the chain $\{0, 1, 2, ..., n-1\}$ of integers).

![Figure 13](image-url)

**Figure 13.** A modular non distributive rectangular lattice $M_3[C_3]$

We prove the following:

**Lemma 3.** Let $G$ be the direct product of chains and let $\Theta$ be a cover-preserving join-congruence of $G$. Then $G/\Theta$ is a rectangular lattice with the property $J(G) \equiv J(G/\Theta)$ if and only if every join-irreducible element of $G$ is a one-element $\Theta$-class.

**Proof.** Let $g/\Theta$ be a $\Theta$-class containing $g \in G$. Assume that $g$ is a minimal element of this class. If $g/\Theta$ is a join-irreducible element of $G/\Theta$ then $g$ is a join-irreducible
element of \( G \), i.e. \( g = (0, \ldots, 0, g_i, 0, \ldots, 0) \) for some \( i \). But \( J(G) \cong J(G/\Theta) \) and therefore \( g/\Theta \) must be a one-element class. The converse is trivial, \( J(G/\Theta) \) is order isomorphic to the paset \( \{(0, \ldots, 0, g_i, 0, \ldots, 0)\} \), i.e. \( G/\Theta \) is rectangular.

The modular lattices \( M_n, n > 2 \) are two dimensional lattices and \( w(J(M_n)) = n \), i.e. \( \dim(M_n) < w(J(M_n)) \).

4. The source

4.1. The source cell and the beret (swiss cup). Let \( \Theta \) be a cover-preserving join-congruence of a finite distributive lattice \( D \).

**Definition 9.** An element \( s \in D \) is called a source element of \( \Theta \) if there is a \( t, t < s \) such that \( s = t \ (\Theta) \) and for every prime quotient \( u/v \) if \( s/t \not\equiv u/v, s \neq u \) imply \( u \neq v \ (\Theta) \). The set \( S_{\Theta} \) of all source elements of \( \Theta \) is the source of \( \Theta \).

Let us remark the source element can defined in an arbitrary semimodular lattice. It is clear that \( \Theta \) is determined by the source \( S_{\Theta} \subset D \).

**Lemma 4.** Let \( x \) be an arbitrary lower cover of a source element \( s \) of \( \Theta \). Then \( x \equiv s \ (\Theta) \). If \( s/x \not\equiv v/z, s \neq v \), then \( v \neq z \ (\Theta) \).

**Proof.** Let \( s \) be a source element of \( \Theta \) then \( s = t \ (\Theta) \) for some \( t, t < s \) and \( x \not\equiv t \) then \( \{x \land t, t, s\} \) form a covering square. Then \( x \not\equiv x \land t \ (\Theta) \). This implies \( x \land t \equiv t \ (\Theta) \). By Lemma 1 we have \( x \equiv s \ (\Theta) \).

To prove that \( v \not\equiv z \ (\Theta) \), we may assume that \( v \not\equiv s \). Take \( t, t < s \), then we have three (pairwise different) lower covers of \( s \), namely \( x, v, t \). These generate an eight-element boolean lattice in which \( s = t \ (\Theta) \), \( s = x \ (\Theta) \) and \( s = v \ (\Theta) \). By the choice of \( t \) we know that \( v \not\equiv x \land t \ (\Theta) \), \( x \not\equiv x \land t \ (\Theta) \) and \( z \not\equiv x \land t \land v \ (\Theta) \). It follows that \( x \not\equiv t \ (\Theta) \), otherwise by the transitivity \( x \not\equiv v \ (\Theta) \). This implies \( t \land x \not\equiv t \land x \land v \ (\Theta) \). Take the covering square \( \{x \land v \land t, z, t \land x, x\} \) then by Lemma 1 \( z \not\equiv x \ (\Theta) \), which implies \( z \not\equiv v \ (\Theta) \).

**Definition 10.** A source cell \( B_s \) is the boolean lattice generated by the lower covers of a source element \( s \in S_{\Theta} \) and \( s_\ast \) denotes the bottom element of the source-cell. The beret \( T_s \) of the source cell \( B_s \) is the unit element, \( s \) and the dual atoms of \( B_s \).

The set of source cells of \( \Theta \) is denoted by \( \text{Sc}(\Theta) \).

If \( s \) is a source element of the cover-preserving join-congruence \( \Theta \), then by Lemma 4 its restriction \( \Psi_s \) to the source cell has only one nontrivial \( \Psi_s \)-class this is the beret. Take the boolean lattice \( C_2^n \) and the cover-preserving join-congruence \( \Theta_n \) with the property that the 1-class is the only one non-trivial congruence class. \( \mathbb{B}_n \) will be denote the factor lattice \( C_2^n / \Theta_n \) (\( \mathbb{B}_2 \cong C_2, \mathbb{B}_3 \cong M_3 \) and \( \mathbb{B}_4 \) on Figure 42). Then we have \( B_s / \Psi_s \cong \mathbb{B}_n \).

The meet of the lower covers of the bottom element \( s_\ast \) of a source cell is denoted by \( s_\ast \ast \).

**Definition 11.** The neighborhood of source cell or a source element \( s \) is the interval \( N_s = [s_\ast \ast, s] \).

On Figure 16 and Figure 17 there is the distributive lattice \( 3^3 \), with a beret as the top of it. A source cell has \( 3^n - 1 \) neighborhood cells and \( N_s \) is isomorphic to \( 3^n \).
**Definition 12.** A source element $s$ of $\Theta$ is called bastard if $s$ itself or at least one of its lower covers $t$ is join-irreducible. The cover-preserving join-congruence $\Theta$ is bastard if has a source element which is bastard.

In the 2-dimensional case this means that $s$ or at least the lower covers of $s$, $a$ and $b$ is join-irreducible.

**Remark.** Let $\varphi : G \to L$ be a cover-preserving join-homomorphism. We denote by $\Theta$ the cover-preserving join-congruence induced by $\varphi$. Take the source $S$ of $\Theta$. A source element $s \in S$ is called bastard if $s$ itself or at least one of its lower covers $t$ is join-irreducible. Let $S' \subseteq S$ the set of all non bastard source elements and $\Theta'$ denotes the corresponding cover-preserving join-congruence. Then $R = G/\Theta'$ is a rectangular lattice (envelop) and $L = G/\Theta$ is a cover-preserving sublattice of $R$.

4.2. **Source lattices.** The join-congruence of $N_s$, where $B_s$ is the only non-trivial congruence class will be denoted by $\Phi_s$.

**Definition 13.** The source lattice is the factor lattice $C_3^n/\Phi_n$, where $\Phi_n$ is the join-congruence with only one non-trivial congruence class containing the unit element and the dual atoms of $C_3^n$.

$L_2 \cong S_7$ and $L_3$ is presented on Figure 11.
Figure 16. A source cell with the neighborhood, $N_s$ and the beret $T_s$.

Figure 17. The factor of $N_s$ in the $Jw(D) = 2$ case, $\mathbb{L}_2 = \mathbb{L}_2^{H}$.

Figure 18. A source cell with the neighborhood and the beret $T_s$.

The skeleton of the source lattice $\mathbb{L}_n$ is the $2^n$ Boole lattice. We defined Kuro-Ore dimension of a lattice $L$ is the minimal number of join-irreducibles needed to span the unit element of $L$. The Kuro-Ore dimension of $\mathbb{L}_n$ is $n$. 
Lemma 5. \( \mathbb{L}_n \) is a subdirect irreducible rectangular semimodular lattice.

Proof. It is an easy exercise, in the J-width = 3 case, Figure 19, \( \text{con}(s, a) \) is the smallest non trivial congruence relation. \( \square \)

4.3. Independence. The source elements are independent in the following sense:

Definition 14. Two elements \( s_1 \) and \( s_2 \) of a distributive lattice are \( s \)-independent if \( x \prec s_1, y \prec s_2 \) then \( s_1/x, s_2/y \) are not perspective, \( s_1/x \not\sim s_2/y \).

Lemma 6. Two elements \( s_1 \) and \( s_2 \) of a distributive lattice are \( s \)-independent if one of the following is satisfied:
1. \( s_1 \) and \( s_2 \) are incomparable,
2. \( s_1 < s_2 \) and \( t \prec s_2 \) implies \( t \geq s_1 \), i.e. \( s_1 \leq s_2^{**} \).

Proof. It is clear that for an incomparable pair \( s_1, s_2 \) if \( u \prec s_1 \) and \( v \prec s_2 \) then \( s_1/u \) and \( s_2/v \) cannot be projective. On the other case, if \( s_1 < s_2 \) then \( t||s_1 \) would imply that \( s_2/t \) and \( s_1/t \wedge s_1 \) are perspective. This means that \( t \geq s_1 \). \( \square \)

Lemma 7. Two source elements of of a cover-preserving join-congruence \( \Theta \) are \( s \)-independent, i.e. if \( s_1 \) and \( s_2 \) are two different source elements of \( \Theta \) and \( x \prec s_1, y \prec s_2 \) then \( s_1/x, s_2/y \) cannot be perspective, \( s_1/x \not\sim s_2/y \).

Proof. This is a trivial consequence of Lemma 4 and Lemma 6. \( \square \)

\( \text{con}^{\text{cp}}(s) \) was defined in 1.3 (see p.7). The source of this cover-preserving join-congruence is \( S = \{ s \} \). Let \( \{ s_1, ..., s_n \} \) be an \( s \)-independent subset of the distributive lattice \( D \).

Lemma 8. \( \Theta = \bigvee \text{con}^{\text{cp}}(s_i) \) is a cover-preserving join-congruence.
Then there is an $i$ such that $a \equiv b \lor c (\Theta)$. Consequently $b \equiv a \lor b (\Theta)$.

\[\text{Lemma 9. Let } s > 0 \text{ be an element of a finite distributive lattice } D. \text{ Then } l(D/\con^\cp(s)) = l(D) - 1.\]

\[\text{Proof. Take a maximal chain } C \text{ of length } n \text{ through } s \text{ and an arbitrary } t \in C, t < s. \text{ By Lemma 4 } t \equiv s \mod \con^\cp(s) \text{ and no other prime interval collapses by } \con^\cp(s). \text{ Therefore the image of } C \text{ in } D/\con^\cp(s) \text{ has length } n - 1. \text{ The lattice } D/\con^\cp(s) \text{ is semimodular (see [15]), which proves the lemma.}\]

\[\text{Theorem 4. Every semimodular lattices } L \text{ can be characterized as a pair } (G, S) \text{ where } G \text{ is the direct power of a chain (grid) and } S \text{ is a s-independent subset of } G.\]

In a boolean lattice s-independent means that $s_1$ and $s_2$ are incomparable, i.e. it is an antichain. As usual, $h(s)$ denotes the height of $s$.

\[\text{Corollary 1. Every geometric lattices } \Theta \text{ is determined by a pair } (B, S), \text{ where } B \text{ is a boolean lattice and } S \text{ is an antichain of } B \text{ with the property } h(s) > 2 \text{ for all } s \in S.\]

Not every s-independent set is the source of a cover-preserving join-congruence.

\[\text{Lemma 10. Let } \Phi \text{ be a join-congruence of a finite semimodular lattice } M. \text{ Then } \Phi \text{ is cover-preserving if and only if } a, b, c \in G \text{ such that } a < a \lor b \lor c, b < a \lor b \lor c, c < a \lor b \lor c, a \equiv a \land b (\Phi), a \equiv a \land c (\Phi), c \equiv b \land c (\Phi), a \land b \not\equiv a \land b \land c (\Phi), a \land b \not\equiv a \land b \land c (\Phi), a \land b \not\equiv a \land b \land c (\Phi) \text{ are satisfied, then } b \equiv a \land b (\Phi), b \equiv b \land c (\Phi).\]

\[\text{Lemma 11. Let } S \text{ be a source of a cover-preserving congruence relation } \Phi \text{ of } G \text{ then } S \text{ satisfies the following property:}\]

\[(3) \text{ if } a, b \in S \text{ and } a < a \lor b, b < a \lor b \text{ then } c \lor a \lor b \text{ then } c \in S.\]

\[\text{Problem 1. Let } D \text{ be a distributive lattice and } a, b, c \in D \text{ such that } a < a \lor b, b < a \lor b, c < a \lor b \lor c. \text{ A subset } S \subseteq D \text{ is the source of a cover-preserving join-congruence if } S \text{ satisfies the conditions:}\]

\[(1) S \text{ is s-independent,}\]
\[(2) (3) \text{ of Lemma 11.}\]

4.4. The matrix of a cover-preserving join-congruence. Let $L$ be a semimodular lattice. By Theorem 1. we have a grid $G = C_k^n$ ($C_k = \{1, 2, \ldots, k\}$ is the chain of natural numbers) and a cover-preserving joint-congruence $\Theta$ of $G$ such that $G/\Theta \cong L$. On Figure 19 we have a 2-dimensional case a matrix. We write into a cover-preserving square an "1" entry if at the bottom we have a source element. Otherwise we write "0". This is a $(0, 1)$-matrix and every row/column contains at most "1" entry. Put 1 into a cell if its top element is in $S$, otherwise put zero. What we get is an $n \times n$ matrix, $M_L$, which determines $L$ (if you like you can turn this grid with 45 degrees to see the matrix in the traditional form).
The first matrix is a $n \times n$ $(0,1)$-matrix, where every row/column contains at most 1 entry:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

There is an other way to define a matrix using the coordinates of the source elements:

\[
\begin{pmatrix}
6 & 5 & 4 & 2 \\
2 & 6 & 3 & 5 \\
\end{pmatrix}
\]

For semimodular lattices $S$ with $\text{dim}(S) \geq 3$ we use hypermatrices. The matrix of $M_3$ is a $(0,1)$-matrix of type $3 \times 3 \times 3$: $[a_{i,j,k}], a_{1,1,1} = 1$ and $a_{i,j,k} = 0$ otherwise.

A column $C(3)_{i,j}$ is $\{a_{i,j,k}; k = 1,2,...,n\}$ of a $3 \times 3 \times 3$ matrix: $[a_{i,j,k}]$ and similarly,

$C(1)_{i,k}$ is $\{a_{i,j,k}; j = 1,2,...,n\}$,

$C(2)_{i,j}$ is $\{a_{j,k}; i = 1,2,...,n\}$.

We use $(0,1)$-matrices, where every column contains at most one entry 1.

PART II

Constructions
Part 2. Constructions

5. CONSTRUCTIONS OF SEMIMODULAR LATTICES

5.1. The $n$-fork construction (the zipper). This general construction was introduced in [6] for slim planar semimodular lattices (semimodular lattices of dimension 2), but it was included in some older papers, see in [14] and [24]. Let $S$ be a 4-cell of a slim semimodular lattice $L$. Then $S$ is a covering square $\{a = b \wedge w, b, w, c = b \vee w\}$, see on Figure 20. We change $L$ to a new lattice $L'$ as follows:

Step 1. Firstly, we replace $S$ by a copy of $S \cong L_2$. This way we get three new 4-cells instead of $S$.

\[ \text{Step 2. Secondly, as long as there is a chain } u \prec v \prec w \text{ such that } v \text{ is a new element and } T = \{x = u \wedge z, z, u, w = z \vee u\} \text{ is a 4-cell in the original lattice } L \text{ but } x \prec z \text{ at the present stage, see Figure 22, we insert a new element } y \text{ such that } x \prec y \prec z \text{ and } y \prec v. \text{ (This way we get two 4-cells instead of } T. \text{) When this “downward-going” procedure terminates, we obtain } L'. \text{ The collection of all new elements, which is a poset, will be called a fork. We say that } L' \text{ is obtained from } L \text{ by adding a fork to } L \text{ (at the 4-cell } S). \text{ See Figure 21 for an illustration. If we add several forks to } L \text{ one by one, then we simply speak of adding forks to } L. \]

**Theorem 5.** [6] Let $L$ be a slim semimodular lattice consisting of at least three elements. Then $L$ can be obtained from the direct product of two nontrivial finite chains such that

- first we add finitely many forks one by one,
- and then we remove corners, one by one, finitely many times.

**Remark.** The second condition means we have a distributive join-congruence on the grid $G$.

On Figure 24, you can see a refinement of a grid $G$, we insert the black filled elements.

Let $S$ be a semimodular lattice and let $G$ be a grid of $S$ with the corresponding cover-preserving join-congruence $\Theta$. If we add a fork to $S$ then this procedure can
be obtained on the following way: we take a refinement of $G$ and select a new source element (which is a new grid element.) See Figure 29-32.
It is easy to generalize this construction for the dimension $n > 2$, we take an $n$-cell which is isomorphic to $2^n$ replace by $L_n$, see Figure 28.

Step 1'. Firstly, we replace a covering cube $S \cong 2^3$ by a copy of $S_24$.

Step 2'. Secondly, as long as there is an other neighbor covering cube we insert to the elements $x, y, z, u, v$ the new elements $x', y', z', u', v'$. When this “downward-going” procedure terminates, we obtain $L'$. The collection of all new elements, which is a poset, will be called a 3-fork. We say that $L'$ is obtained from $L$ by
Figure 26. Refinement II.

Figure 27. The source lattice $L_3 = S_{24}$ and the “downward-going” procedure

adding a fork to $L$, see Figure 20 for an illustration. If we add several forks to $L$ one by one, then we simply speak of adding forks to $L$.

On Figure 24. you can see a 3-fork.
5.2. The pigeonhole. To add a fork we can make the following procedure:

\[ \begin{array}{c}
(0,0) \\
(1,0) \\
(0,0) \\
(1,1) \\
(1/2,0) \\
(1/2,0) \\
(1/2,1/2) \\
(1,0) \\
(0,1/2) \\
(0,0) \\
\end{array} \]

(refinement of the grid) \rightarrow \rightarrow \rightarrow \rightarrow

\[ \begin{array}{c}
(0,0) \\
(1,0) \\
(0,0) \\
(1/2,0) \\
(1/2,1/2) \\
(1,0) \\
(0,1/2) \\
(0,0) \\
\end{array} \]

add (nesting) a fork

Figure 28. Figure 47 The iterative procedure

5.2.1. The \( \text{Jw}(L) = 2 \) case. We define the nested slim lattices inductively, these are lattices where copies of \( S_7 \)-s are nested into each other:

Definition 15. A slim semimodular lattice \( N \) is called a 2-nested lattice if we obtain with the following iterative procedure

1. The four element Boolean lattice, i.e. the 4-cell \( C_2^2 \), this is a 4-cell,
2. We add a fork to this 4-cell \( C_2^2 \), we obtain the source lattice \( S_7 = L_2 \) (on Figure 30 (III) or (V)), and apply the "downward-going" procedure if it is necessary (this two steps gives eight different 2-nested lattices),
3. \( S_7 \) has three 4-cells. To some of these 4-cells we add further forks, i.e. we replace some covering square by a copy of \( S_7 \),
4. We continue this procedure in finite many steps (on Figure 30 (II)).

The set of all 2-nested lattices will be denoted by \( \mathcal{R}_1 \).

Theorem 6. Every 2-nested lattice \( N \) satisfies the following properties:

1. \( N \) is rectangular,
2. \( N \) has a complementary pair \( a, a' \) (\( a \land a' = 0 \) and \( a \lor a' = 1 \)) such that \( a \prec 1 \) and \( a' \prec 1 \),
3. \( N \) contains as ideal two chains \( C_1 \) and \( C_2 \), such that \( C = C_1 \cup C_2 \) is congruence-determining.

Proof. \( \square \)
On the grid $G$ there is a natural distance, $\delta(a, b)$ of two elements $a, b$ in a grid $G = C^m$ is the smallest natural number $n$ such that there is a sequence $a = c_1, c_2, ..., c_{n-1} = b$, $c_i < c_{i+1}$ or $c_i > c_{i+1}$ for every $i$. This defines an equivalence relation $\mathcal{E}$ on $S$. Two source elements are of the same block iff their are "close to each other".

**Definition 16.** Two incomparable elements $a$ and $b$ of a distributive lattice $D$ are horizontal-adjacent if $\delta(a, b) = 2$, i.e. if $\{a \wedge b, a, a \vee b\}$ is a covering square. Two elements $a$ and $b$, $b < a$ of a distributive lattice $D$ are vertical-adjacent $b = a^{**}$.

**Definition 17.** $a \equiv b$ ($\mathcal{E}$) if and only if there is a sequence $a = s_1, s_2, ..., s_n = b$ of source elements in a grid $G$ such that for every $i$ $(s_i, s_{i+1})$ is either a horizontal-adjacent pair or a vertical-adjacent pair.

**Definition 18.** $\{a, b\}$ is a remote pair if either (1) $a$ and $b$ are incomparable and $\delta(a, b) > 2$ or (2) $b < a^{**}$.

**Definition 19.** A The element $s$ isolated in respect of $S$ if $\delta(s, t) > 2$ for every $t \in S, t \neq s$.

**Lemma 12.** $\mathcal{E}$ is an equivalence relation on the source $S$.

**Proof.**

Let $A$ be a block of $\mathcal{E}$, this is a subset of the source $S$. Take the interval $\overline{A}$ of $G$ generated by $A$. We prove that the restriction of $S$ to $A$ defines a nested lattice $N_A$.
5.2.2. Some nested lattices. Consider the following semimodular lattices:

Let \( n \) and \( m \) be natural numbers and let \( C = \{0, 1, ..., n-1\} \) be a chain of length \( n-1 \). Then

\[
L^{m,n}_H = \text{the meet-sublattice of } C^m \text{ consisting all elements } (x_1, ..., x_m) \text{ where either } x_1 + x_2 + ... + x_m \leq n - 1 \text{ or } x_1 = ... = x_m = n - 1.
\]

The elements \( s_i = (x_1, ..., x_m) \), \( x_1 + x_2 + ... + x_m = n - 1 \) form an s-independent set \( S \) in \( G \). This is a vertically adjacent set. The lattice \( L^{m,n}_H \) is the factor lattice \( G/\Theta_S \). Obviously, this is a rectangular lattice of J-width \( m \). See the examples on Figure 45, Figure 47, Figure 49, Figure 54.
\( L^V_{2,1} \) is the lattice \( S_7 \). It is easy to see that \( L^V_{2,2} \) is determined by an horizontal-adjacent independent set \( S \), which contains the elements \( (x_1, \ldots, x_m) \) where \( x_1 = \ldots = x_m \geq 2 \) (see Figure 22.) Take again \( G = C^m \), where \( C = \{0, 1, \ldots, n-1\} \). We define the vertical crocheted lattice

\[
L^V_{m,n}
\]

(see Figure 23). This is the sublattice of \( C^m \) which contains all elements in the form \((0, \ldots, 0, i, 0, \ldots, 0)\) and the diagonal elements \((j, j, \ldots, j)\) for \( j > 2 \).

Figure 31. The nested lattice \( L^H_{2,7} \) with the skeleton

5.2.3. The \( Jw(L) = 3 \) case. We define the 3-nested lattices similarly as in the the \( Jw(L) = 2 \) case. We start with the \( C_2^3 \) Boolean lattice, this is a 3-nested lattice. We add a 3-fork to this \( 2^3 \)-cell, we obtain the source lattice \( L_3 \), see on Figure 27 and apply the ”downward-going” procedure if it is necessary, see on Figure 24. This is a 3-nested lattice. \( L_3 = S_{24} \) contains 7 \( 2^3 \)-cells. In some of these we insert 3-forks. All these are 3-nested lattices. If we continue this procedure we get all the 3-nested lattices.

5.3. The patchwork system. \cite{26},\cite{27} Matching is a special case of S-glued sum. The S-glued sum was introduced by Christian Herrmann \cite{21}.

**Definition 20.** (S-glued system). Let \( S \) and \( L_s, s \in S \), be lattices of finite length. The system \( L_s, s \in S \) is called an S-glued system iff if the following conditions are satisfied:

1. For all \( s, t \in S \), if \( s \leq t \), then either \( L_s \cap L_t = \emptyset \) or \( L_s \cap L_t \) is a filter in \( L_s \) and an ideal in \( L_t \).
2. For all \( s, t \in S \) with \( s \leq t \) and for all \( a, b \in L_a \cap L_b \), the relation \( a \leq b \) holds in \( L_s \) iff \( a \leq b \) in \( L_t \).
3. For all \( s, t \in S \), the covering \( s \prec t \) implies that \( L_s \cap L_t \neq \emptyset \).
4. If \( s, t \in S \), then \( L_s \cap L_t \subseteq L_{s \land t} \cap L_{s \lor t} \).

**Definition 21.** (S-glued sum) Let \( L = \bigcup(L_s | s \in S) \), where \( L_s, s \in S \) is an S-glued system. Let the partial order \( \leq \) in \( L \) is defined as follows: for \( a, b \in L \), let \( a \leq b \) iff there exists a sequence \( a = x_0, x_1, \ldots, x_n = b \) of elements of \( L \) and a sequence \( s_0, \ldots, s_n \) of elements of \( S \) such that \( s_i \leq s_{i+1} \) in \( S \), \( i = 1, \ldots, n-1 \), and \( x_{i-1} \leq x_i \) in \( L(s_i), i = 1, \ldots, n \). Then \( L \) is a lattice, the S-glued sum of \( L_s, s \in S \).
We call the \( L_s \) components of the blocks of the S-glued sum. Any block is an interval in \( L \). In this paper we will use as blocks special rectangular lattices.

Ch. Herrmann proved that every modular lattice \( L \) of finite length is the S-glued sum of its maximal complemented intervals (these are, obviously rectangular lattices).

In section 1.2 there are two examples. In the kitchen the mosaics (which are two dimensional) are glued together by an edge (one dimensional). The bricks (three dimensional) are glued together by a side (two dimensional).

**Definition 22.** (Matching) Let \( S \) and \( L_s, s \in S \), be semimodular lattices of finite length. The S-glued sum \( L = \bigcup \{ L_s \mid s \in S \} \), where \( L_s, s \in S \) is called of \( L_s \)-s if

\[
\text{Jw}(L_s \cap L_t) < \min(\text{Jw}(L_s), \text{Jw}(L_t)).
\]

Every planar distributive lattice is the matching of its covering squares and some of the edges (narrrows), see Figure 2. For cubes, see Figure 4 and Figure 5.

5.4. **The planar semimodular lattices.** First, we consider the slim planar semimodular lattices (The slim planar semimodular lattice was defined in [15], these are planar lattices where the covering squares are intervals).

**Theorem 7.** Every slim planar semimodular lattice is the matching of filters of nested lattices.

**Proof.** Induction on the length. Take the equivales relation \( \mathcal{E} \). \( \square \)

5.5. **The patchwork in higher dimensions.**

**Theorem 8.** Let \( L \) be a semimodular lattice of J-width \( n > 2 \). Then \( L \) is the matching of filters of nested lattices.

**Proof.** Similar to the proof of Theorem 6. \( \square \)
Revisted: December 13, 2012
PART III

Structure theorems
Part 3. Structure theorems

6. Rectangular lattices as packing boxes and building stones.

6.1. The rectangular hulls of semimodular lattices.

Theorem 9. Every semimodular lattice \( L \) has a rectangular extension \( R \) such that

1. \( L \) and \( R \) have the same length, i.e. \( L \) is a cover-preserving \((0,1)\)-sublattice of \( R \);
2. \( L \) and \( R \) have the same dimension, \( \dim(L) = \dim(R) \) (i.e. \( J(L) \) and \( J(R) \) have the same width).

\( R \) is called a rectangular hull of \( L \) (or packing box).

Proof. Let \( L \) be a semimodular lattice of length \( n \) and \( Jw(L) = k \). By Theorem 1, \( L \) is the cover-preserving join-homomorphic image of the distributive lattice \( F \). Let \( \Phi \) be the corresponding cover-preserving join-congruence, \( S \) denotes the source of \( \Phi \). On the other hand, by Lemma 1, \( F \) is the cover-preserving sublattice of the grid \( G \). We extend \( \Phi \) to \( G \). Let \( a/b \) and \( c/d \) prime quotients of a lattice \( L \). If \( b \lor c = a \), \( b \land c = d \) we say that \( a/b \) is perspective up to \( c/d \) and we write \( a/b \searrow c/d \). This cover-preserving join-congruence will be denoted by \( \Phi \) and is defined as follows: for a covering pair \( d \prec c \), \( c,d \in G \), \( c \equiv d \) (\( \Phi \)) if and only if there is a source element \( s \in S \subset F \) a lower cover \( u \in F \) of \( s \) such that \( c/d \) is perspective up to \( s/u \). It is easy to prove that the transitive extension is a cover-preserving join-congruence.

Then define \( R \) as:

\[
R = G/\Phi.
\]

By the second isomorphism theorem we have the following isomorphisms and join-homomorphisms, presented on the diagram:

\[
\begin{align*}
L &\cong F/\Phi \cong (G/\Psi)/(\psi \lor \Phi)/\Psi \cong G/(\Psi \lor \Phi) \cong (G/\Phi)/((\psi \lor \Phi)/\Phi) \cong R/\Psi,
\end{align*}
\]

\[
\begin{array}{ccc}
F & \xrightarrow{\psi} & L \\
\uparrow\psi & & \uparrow\psi \\
G & \xrightarrow{\overline{\psi}} & R
\end{array}
\]

Then \( c = d \lor s \), which means that \( c \) is reducible in \( G \). Assume that \( d \) is a join-irreducible element of \( G \). Every element of the ideal \( d \) is join-irreducible, i.e. \( u \in F \) must be join-irreducible too. This means the join-irreducible element of \( F \) is in a non-trivial \( \Phi \)-class, in contradiction to Lemma 2. This proves that every join-irreducible element of \( G \) is a one-element \( \Phi \)-class, i.e. \( R \) is a rectangular lattice.

\( \Phi \) and \( \overline{\Phi} \) by Lemma 3, \( F \) and \( G \) have the same length, i.e. \( L \) is a cover-preserving sublattice of \( R \).

By the definition of \( G \) it is trivial that \( J(L) \) and \( J(R) \) have the same width. \( \square \)
6.2. The patchwork system. Matching is a special case of S-glued sum. The S-glued sum was introduced by Christian Herrmann [21].

Definition 23. (S-glued system). Let \( S \) and \( L_s, s \in S \), be lattices of finite length. The system \( L_s, s \in S \) is called an S-glued system iff if the following conditions are satisfied:

1. For all \( s, t \in S \), if \( s \leq t \), then either \( L_s \cap L_t = \emptyset \) or \( L_s \cap L_t \) is a filter in \( L_s \) and an ideal in \( L_t \).
2. For all \( s, t \in S \) with \( s \leq t \) and for all \( a, b \in L_a \cap L_b \), the relation \( a \leq b \) holds in \( L_s \) iff \( a \leq b \) in \( L_t \).
3. For all \( s, t \in S \), the covering \( s \prec t \) implies that \( L_s \cap L_t \neq \emptyset \).
4. If \( s, t \in S \), then \( L_s \cap L_t \subseteq L_s \wedge t \cap L_s \vee t \).

Definition 24. (S-glued sum) Let \( L = \bigcup (L_s | s \in S) \), where \( L_s, s \in S \) is an S-glued system. Let the partial order \( \leq \) in \( L \) is defined as follows: for \( a, b \in L \), let \( a \leq b \) iff there exists a sequence \( a = x_0, x_1, ..., x_n = b \) of elements of \( L \) and a sequence \( s_0, ..., s_n \) of elements of \( S \) such that \( s_i \leq s_{i+1} \) in \( S, i = 1, ..., n - 1 \), and \( x_{i+1} \leq x_i \) in \( L_{s_{i+1}}, i = 1, ..., n \). Then \( L \) is a lattice, the S-glued sum of \( L_s, s \in S \).

We call the \( L_s \) components of the blocks of the S-glued sum. Any block is an interval in \( L \). In this paper we will use as blocks special rectangular lattices.

Ch. Herrmann proved that every modular lattice \( L \) of finite length is the S-glued sum of its maximal complemented intervals (these are, obviously rectangular lattices).

In section 1.2 there are two examples. In the kitchen the mosaics (which are two dimensional) are glued together by an edge (one dimensional). The bricks (three dimensional) are glued together by a side (two dimensional).

Definition 25. (Matching) Let \( S \) and \( L_s, s \in S \), be semimodular lattices of finite length. The S-glued sum \( L = \bigcup (L_s | s \in S) \), where \( L_s, s \in S \) is called of \( L_s \)-s if \( \dim(L_s \cap L_t) < \min(\dim(L_s), \dim(L_t)) \).

Every planar distributive lattice is the matching of its covering squares and some of the edges (narrow), see Figure 2. For cubes, see Figure 4 and Figure 5.

6.3. The planar semimodular lattices. First, we consider the slim planar semimodular lattices (The slim planar semimodular lattice was defined in [15], these are planar lattices where the covering squares are intervals).

Theorem 10. Every slim planar semimodular lattice is the matching of filters of nested lattices.

Proof. Induction on the length. Take the equivalence relation \( E \). \( \square \)

6.4. The patchwork in higher dimensions.

Theorem 11. Let \( L \) be a semimodular lattice of dimension \( n > 2 \). Then \( L \) is the matching of filters of nested lattices.

Proof. Similar to the proof of Theorem 6. \( \square \)

6.5. Special embeddings.
6.5.1. Congruence-preserving embedding. Let $B$ be a sublattice of a lattice $A$. If $0_A, 1_A \in B$, then $B$ is said to be a $(0,1)$-sublattice. If every congruence of $A$ is determined by its restriction to $B$, then $B$ is called a congruence-determining sublattice of $A$. Ideals that are chains will be called chain ideals. Consider the class $\mathcal{K} = \{L : L$ is a finite length semimodular lattice that has a congruence-determining chain ideal}.

Theorem 12. Let $L \in \mathcal{K}$, and let $D$ be a $(0,1)$-sublattice of $\text{Con}L$. Then there exists an $\overline{L} \in \mathcal{K}$ such that the restriction mapping $\rho : \text{Con}\overline{L} \to \text{Con}L, \theta \mapsto \theta|_L$, is actually a $(0,1)$-lattice isomorphism $\text{Con}\overline{L} \to D$; in particular, $\text{Con}\overline{L} \cong D$.

Theorem 13. Every planar semimodular lattice $L$ has a congruence-preserving extension $K$ such that

1. $K$ is a planar semimodular lattice,
2. $L$ is an almost-filter of $K$ (it contains $1$ and is a cover-preserving sublattice)
3. $K$ contains as ideal a chain $C$,
4. $C$ is congruence-determining.

6.5.2. Embedding into geometric lattice. It was proved by G Grätzer and E. W. Kiss [13], (for semimodular lattices of finite length see G. Czédli and E. T. Schmidt [5]):

Theorem 14. Every finite semimodular lattice $L$ has a cover-preserving embedding into a geometric lattice $G$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure33.png}
\caption{$\mathbb{B}[(4,3),(3,4),(2,2)]$}
\end{figure}
7. Modulararity

7.1. Skeleton. G. Grätzer and R. W. Quackenbush proved in [20] that a planar distributive lattice $D$ with more than two elements is isomorphic to the Skeleton($L$), for a subdirectly irreducible planar modular lattice $L$ iff $L$ has no narrows. The results Theorem 4.9, Theorem 5.2 can be generalized for arbitrary finite order-dimension.

We prove that this statement is true for an arbitrary finite distributive lattice $D$ with $\text{Jw}(D) < \aleph_0$ (defined in 1.1).

**Lemma 13.** A modular lattice $L$ which contains a prime interval $p$ is subdirectly irreducible if and only if $L$ is weakly atomic and any two prime intervals are projective.

**Proof.** It is trivial that a modular lattice $L$ which contains a prime interval $p$ and satisfies the given conditions is subdirectly irreducible.

Let $L$ be a subdirectly irreducible modular lattice $L$ which contains a prime interval $p$. We may assume that for any $a < b$

$$\text{con}(p) \leq \text{con}(a, b)$$

. By the weakly modularity ([19], p.194) the interval $[a, b]$ contains a prime interval $q$ such that $p$ and $q$ are projective, i.e. $L$ is weakly atomic. \hfill $\Box$

**Theorem 15.** Let $D$ be a distributive lattice with more than two elements, $\dim(D) < \aleph_0$. Then $D$ is isomorphic to Skeleton($L$), for a subdirectly irreducible modular lattice $L$, iff $D$ is narrows free.

**Theorem 16.** There exists a subdirect irreducible modular lattice of order-dimension $\aleph_0$ which does not contain a prime interval.

**Proof.** $\mathbb{Q}$ is the $[0, 1]$ chain of rational numbers. Glue together copies of $M_3[\mathbb{Q}]$-s, see E. T. Schmidt [23]. \hfill $\Box$

7.2. Diamond-free semimodular lattices.

7.3. Modular source.

**Definition 26.** A source $S$ of a distributive lattice $D$ is called modular if the lattice $D/\Theta_S$ is modular.

**Lemma 14.** Let $s$ be a source element of a grid $G$. Then $G/\Theta_s$ is modular lattice iff $s$ is bastard.

On the following two figures (Figure 30 and Figure 31) you can see the representation of $M_3$ with the grid resp. lower grid.

Let $D$ be a planar distributive lattice. The J-width is two. Add doubly-irreducible elements to the interiors of some 4-cells (covering squares) you get a planar modular lattice $M$. If we start with a distributive lattice $D$ of J-width 3 then we can extend some of the covering-cubes into modular non-distributive lattice. $D$ is the skeleton.

Let $\Theta$ be a cover-preserving join-congruence of a grid $G$. Give a necessary and sufficient condition to be $G/\Theta$ modular.

Take the finite field $\mathbb{G}F(p^n)$, $p = 2, n = 1$. The corresponding two-dimensional projective geometry $F = \mathbb{G}P_2$ is the Fano plane. The one-dimensional lattice $\mathbb{G}P_1$ is $M_3$. It is clear that the J-width of $F$ is 7.
On the next picture you can see the "traditional" presentation of the subspace lattice.

Draw the diagram a little bit others we get the following diagram for the same lattice (the same, but differently).

Here we see a cube (fat lines, the skeleton) and six circles on the faces of this cube and two circles (yellow) are inside the cube. If $D$ is the direct product of three
Figure 37. The "traditional" diagram of the subspace lattice of the Fano plane

Figure 38. The subspace lattice of the Fano plane and the Kuroš-Ore skeleton

chains then this contains unit (covering) cubes. We can extend $D$ if we put the Fano plane into some covering cubes this is the skeleton. (Fano plane "locked" in a cube.) By plain lattices we extend a covering square to an $M_3$.

8. The Grätzer-Kiss theorem

It was proved by G. Grätzer and E. W. Kiss [13]:

**Theorem.** Every finite semimodular lattice $L$ has a cover-preserving embedding into a geometric lattice $G$. 
9. Reduction of $J(L)$

We delete some edges of a finite poset $P$, we get the poset $Q$. The poset $P$ is called an refinement of $Q$ or we say $Q$ is an reduction (or pruning) of $P$. Let $L$ be a finite semimodular lattice and take $P = J(L)$.

Does there exit a semimodular lattice $K$ which satisfies the following properties:

1. $J(K) \cong Q$,
2. $L$ is a cover-preserving sublattice of $K$?

The answer is no. (See Figure 31.)

![Diagram](image)

Here is a counter example (given by G. Czédli). Let $L$ be a finite semimodular lattice and let $Q$ be a reduction of $P = J(L)$. For which $Q$ has a cover-preserving embedding into a finite semimodular lattice $K$ with the property $J(K) \cong Q$.

See $L$, $P$ and $Q$ given in the Figure 7. Now $L$ is semimodular. $P = J(L)$. We obtain $Q$ by deleting an edge from the diagram of $P$, so $Q$ is a refinement of $P$. But there is no semimodular lattice $K$ such that $J(K) = Q$ and $L$ is a cover-preserving sublattice of $K$. To show this, suppose the contrary. We may suppose that $l(K) = l(L)$ for otherwise $K$ can be replaced with an interval of $K$. Clearly, $r$ and $s$ (as minimal join-irreducible elements) are atoms in $K$. By semimodularity the height $h(r \lor s) = 2$ in $K$. Since $J(K) = Q$, we have $r \lor s \leq q$ in $K$. We cannot have equality here, for $q \in J(K) = Q$. Hence, $l(K) = 3 \leq h(q)$, implying $q = 1_K \geq p$ in $K$. This contradicts $J(K) = Q$.

10. Appendix (Picture Gallery)

References

[1] K. Baker, z


[22] J. B. Nation, c


[26] [122], E. T. Schmidt, Play with matrices get a structure theorem for semimodular lattices,

[27] [123], E. T. Schmidt, A structure theorem of semimodular lattices: the patchwork representation,