1. Rectangular lattices

Rectangular lattices were introduced by Grätzer-Knapp [3] for planar semimodular lattices. This notion is an important tool by the description of planar semimodular lattices. \( J(L) \) denotes the order of all nonzero join-irreducible elements of \( L \) and \( J_0(K) \) is \( J(L) \cup 0 \).

Let \( X, Y \) be posets. The disjoint sum \( X + Y \) of \( X \) and \( Y \) is the set of all elements in \( X \) and \( Y \) considered as disjoint. The relation \( \leq \) keeps its meaning in \( X \) and in \( Y \), while neither \( x \geq y \) nor \( x \leq y \) for all \( x \in X, y \in Y \).

If \( R \) is a rectangular slim semimodular lattice then \( J(R) \) is the disjoint sum of two chains \( C_1 \) and \( C_2 \) which means that that \( x \in C_1, x \in C_2 \) are incomparable. The width \( w(P) \) of a (finite) order \( P \) is defined to be \( \max\{n: P \text{ has an } n \text{-element antichain}\} \). The width of \( J(L) \) is called the dimension of a semimodular lattice \( L \) and will be denoted by \( \text{dim}(R) \). An other dimension concept is \( \text{Dim}(L) \). \( n = \text{Dim}(L) \) is the greatest integer such that \( L \) contains a sublattice isomorphic to the \( 2^n \)-element boolean lattice. If \( L \) is a distributive lattice then \( \text{dim}(L) = \text{Dim}(L) \). On the other hand \( \text{Dim}(M_3) = 2 \) and \( \text{dim}(M_3) = 3 \).

\( C_n \) denotes an \( n \)-element chain. By G. Czédli, E. T. Schmidt, [2] \( \text{dim}(L) = 3 \) is equivalent to the condition: there 3 disjoint chains \( C_n, C_m \) and \( C_k \), \( J(R) = C_n \cup C_m \cup C_k, n \leq m \leq k \) such that \( R \) is the cover-preserving join-homomorphism of \( G = G_R = C_n \times C_m \times C_k \). In this case we say that \( R \) is of type \( (n, m, k) \). Two dimensional semimodular lattices are the slim lattices. \( G_R \) is called the (lower) grid of \( R \).

\textbf{Remark:} the upper grid of a semimodular lattice \( L \) is \( \overline{G} = C^3 \), where \( C \) is a chain which has the same length as \( L \). By [2] \( L \) is the cover-preserving join-homomorphic image of \( \overline{G} \).

Regularity can be defined for arbitrary dimension:

\textbf{Definition 1.} A rectangular lattice \( L \) is a finite semimodular lattice in which \( J(L) \) is the disjoint sum of chains.

If you has a slim rectangular lattice \( R \) visually, this looks like to Figure 1, i.e. – properly drawn – we see the contour in Figure 2.

If you has a planar semimodular lattice and if you draw properly then you get Figure 2, in the non slim case the dimension is grater then 2, the lattice in Figure 2 has dimensión 5.

Let \( R \) be a 3-dimensional rectangular semimodular lattice. If we have 3 disjoint chains \( C_n, C_m \) and \( C_k \) then rectangular means that \( J(R) \) is the disjoint sum of...
these chains. How does it looks like $R$? The first answer is, visually we see Figure 3, if you draw "properly". The direct product $G = C_n \times C_m \times C_k$ is such a lattice, which looks like to Figure 3. There is an other lattice of this type: this is $M_3[C_n]$, see Figure 5 (here as patchwork of covering squares and $M_3$-s). It is interesting that this is modular.

The expression "it looks like" is not an exact property, to define this exactly we introduce the following concept:
Definition 2. The skeleton of a 3D semimodular lattice is an eight-element boolean lattice which contains 0 and 1.

The skeleton of $L$ will be denoted by $\text{Sk}(L)$. The skeleton of a 2D semimodular lattice is a four-element boolean lattice which contains 0 and 1.

That the 3-dimensional lattice $R$ looks like to Figure 3 means that $R$ contains a skeleton, in this case this means that $\text{dim}(R) = \text{Dim}(R)$. It is easy to see that $\text{Sk}(R) = \text{Sk}(G_R) = \text{Sk}(\overline{G_R})$. In this paper we would like to describe the 3-dimensional rectangular lattices especially those which don’t contains a skeleton.

2. EXAMPLES

We consider first, some exemplars. The simplest case is that $J(L)$ is the 3-element antichain, i.e. $L$ is join-generated by the following order $P$, see in Figure 6;

Then we have a lattice of type $(1, 1, 1)$. Take the order $P$ in Figure 6 this is a chopped lattice, where $a \land b = a \land c = b \land c = 0$. The semimodular lattices join-generated by $P$ are the eight-element boolean lattice and $M_3$ the diamond.

If $L$ is of type $(2, 2, 2)$ the we have the order $Q$, see in Figure 7. The cover-preserving join-homomorphic images of $C_2^3$ (with more then 2 elements) are the eight-element boolean lattice and $M_3$. 

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**Figure 4.** Improperly drawn 8-element boolean lattice.

**Figure 5.** $M_3[C_4]$ as patchwork.
Then $Q$ is a chopped lattice which join generates in the class of semimodular lattices either $C_3^3$ this is a cube or $M_3[C_3]$.

**Remark.** This is equivalent to the following: $C_2^3$ has only one non trivial cover-preserving join-congruence, which is where the dual atoms with 1 form a congruence class.

These generates either $C_2^3$ or $M_3[C_3]$.

$M_3[C_3]$ is a cover-preserving join-homomorphic image of $C_2^3$. You can see this lattice in the cover of Algebra Universalis. On the home page of E. T. Schmidt there is a rotary example.

It follows that we can consider $M_3[C_3]$ as the "modular cube".

$C_2^3$ and $M_3[C_3]$- as geometric shapes have 6–6 flaps.

In Figure 5. we have a 3-dimensional patchwork of 8 monochromatic cubes.

3. **The source**

To describe the cover-preserving join-congruences of $C_n^3$ we need the notion of source elements.

Let $\Theta$ be a cover-preserving join-congruence of a distributive lattice $G$ (which is not necessarily the grid).
Figure 8. A semimodular lattice of type \((2, 2, 2)\) with \(J_0(K)\).

**Definition 3.** An element \(s \in G\) is called a source element of \(\Theta\) if there is a \(t, t \prec s\) such that \(s \equiv t\ (\Theta)\) and for every prime quotient \(u/v\) if \(s/t \not\downarrow u/v, s \neq u\) imply \(u \neq v\ (\Theta)\). The set \(S_\Theta\) of all source elements of \(\Theta\) is the source of \(\Theta\).

**Lemma 1.** Let \(x\) be an arbitrary lower cover of a source element \(s\) of \(\Theta\). Then \(x \equiv s\ (\Theta)\). If \(s/x \not\downarrow v/z, s \neq v\), then \(v \neq z\ (\Theta)\).

**Proof.** Let \(s\) be a source element of \(\Theta\) then \(s \equiv t\ (\Theta)\) for some \(t, t \prec s\). If \(x \prec s\) and \(x \neq t\) then \(\{x \wedge t, x, t, s\}\) form a covering square. Then \(x \neq x \wedge t\ (\Theta)\). This implies \(x \wedge t \neq t\ (\Theta)\). By Lemma 1 we have \(x \equiv s\ (\Theta)\).

To prove that \(v \neq z\ (\Theta)\), we may assume that \(v \prec s\). Take \(t, t \prec s\), then we have three (pairwise different) lower covers of \(s\), namely \(x, v, t\). These generate an eight-element boolean lattice in which \(s \equiv t\ (\Theta)\), \(s \equiv x\ (\Theta)\) and \(s \equiv v\ (\Theta)\). By the choice of \(t\) we know that \(v \neq v \wedge t\ (\Theta)\), \(x \neq x \wedge t\ (\Theta)\) and \(z \neq x \wedge t \wedge v\ (\Theta)\). It follows that \(x \neq t\ (\Theta)\), otherwise by the transitivity \(x \neq v\ (\Theta)\). This implies \(t \wedge x \neq t \wedge x \wedge v\ (\Theta)\). Take the covering square \(\{x \vee v \wedge t, z, t \wedge x, x\}\) then by Lemma 1 \(z \neq x\ (\Theta)\), which implies \(z \neq v\ (\Theta)\). \(\square\)

The following results are proved in [4]. The source \(S\) satisfies an independence property:

**Definition 4.** Two elements \(s_1\) and \(s_2\) of a distributive lattice are \(s\)-independent if \(x \prec s_1, y \prec s_2\) then \(s_1/x, s_2/y\) are not perspective, \(s_1/x \not\sim s_2/y\). A subset \(S\) is \(s\)-independent iff every pair \(\{s_1, s_2\}\) is \(s\)-independent.

\[G = C_n \times C_m \times C_k\] can be considered as a 3D hypermatrix, this has a row and two columns. \(G\) contains covering cubes, these are called cells. The source elements are top element of the sources., see Figure 9.

**Lemma 2.** Every row/column contains at most one source element.
Lemma 3. Two elements \( s_1 \) and \( s_2 \) of a distributive lattice are \( s \)-independent if one of the following is satisfied:

1. \( s_1 \) and \( s_2 \) are incomparable,
2. \( s_1 < s_2 \) and \( t < s_2 \) implies \( t \geq s_1 \), i.e. \( s_1 \leq s_2^* \).

Proof. It is clear that for an incomparable pair \( s_1, s_2 \) if \( u \prec s_1 \) and \( v \prec s_2 \) then \( s_1/u \) and \( s_2/v \) cannot be projective. On the other case, if \( s_1 < s_2 \) then \( t|s_1 \) would imply that \( s_2/t \) and \( s_1/t \wedge s_1 \) are perspective. This means that \( t \geq s_1 \). \( \square \)

It is easy to prove that every \( s \)-independent subset \( S \) generate a cover-preserving join-congruence \( \Theta \). The semimodular lattice \( L \) is characterized by \((G, \Theta)\) or \((G, S)\), where \( S \) is an \( s \)-independent subset. We write:

\[
L = \mathcal{L}(G, S).
\]

Theorem. A rectangular 3D semimodular lattice \( R \) of type \((n, m, k)\) has a 3-skeleton if and only if the source \( S \) of \( R \) has less then \( n \) elements.

Proof. Let \( R \) be a rectangular 3D semimodular lattice of type \((n, m, k)\), \((n \leq m \leq k)\).

Then we have the grid \( G = C_n \times C_m \times C_k \) a cover-preserving join-congruence \( \Theta \) and the source \( S \) of \( \Theta \) such that \( R \cong G/\Theta \). Then \( G \) has a skeleton \( \text{Sk}(G) = \{0, a, b, c, p, q, r, 1\} \), see in Figure 10 (in this example the type is of type \((2, 2, 2)\)). Take the ideal generated by \( a = (c_n, 0, 0) \) of \( G \) (in Figure 10 the yellow line, this is the leading line). \( [(c_{n-1}, 0, 0), (c_n, c_m, c_k)] \) is the first row of the “matrix” \( G \).

If every row contains a source element, i.e. we have \( n \) source elements then \( p \equiv 1(\Theta) \). This implies that \( \text{Sk}(G)/\Theta \) is not a boolean lattice, i.e. \( R \) has no skeleton.

Assume that we have rows without source elements. then \( p \not\equiv 1(\Theta) \). Then the image of \( \text{Sk}(G) \) by the cover-preserving join-homomorphism \( \varphi : G \to R \) is the skeleton \( \text{Sk}(R) \) of \( R \). \( \square \)
Take $M_3$, then $J(M_3)$ is the three-element antichain, i.e. $G = C_2^3$. $G$ has only one cell, the unit element is a source element. Let $\Theta$ be the corresponding cover-preserving join-congruence. The face lattice is $G/\Theta = M_3$. If $R = N_7$ then the grid is $G = C_3^2$ the matrix is not invertible and $R$ and $R$ has a skeleton.

![Figure 10. The skeleton of a $G$.](image)

4. SOME MORE EXAMPLES

Example 1.
We would like to describe all lattices of type $(1, 2, 2)$. If $R$ is such a lattice, then the grid is $G = C_2 \times C_3 \times C_3$. This has 4 cells, one row and two columns.

![Figure 11. The skeleton $G = C_2 \times C_3 \times C_3$.](image)

Example 2.

5. IMPORTANT RECTANGULAR LATTICES: PATCH LATTICES

5.1. Patch lattices. Let $R$ be a 3D semimodular lattice of type $(n, n, n)$.

$R$ is called a patch lattice if in every row and column $s$ except the last row and last (two) columns is a source element. The patch lattices were introduced in [2].
for the two dimensional semimodular lattices, these are the building stones. Every path lattice $R$ has a skeleton $\text{Sk}(R)$. The dual atoms of $\text{Sk}(R)$ are dual atoms of $R$.

5.2. The matrix representation. Take the grid $G$ of $R$. This can be considered as a hypermatrix. If the cell is labeled by a source element then we write as entry 1 into this cell. otherwise we write 0. then we have a $(0,1)$-hypermatrix, see Figure 12.

From Theorem 1 it follows that the condition $R$ has no skeleton is equivalent to the condition ”the hypermatrix is invertible”.

References


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