RECTANGULAR LATTICES AS GEOMETRIC SHAPES

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1. Rectangular lattices

Rectangular lattices were introduced by Grätzer-Knapp [3] for planar semimodular lattices. This notion is an important tool by the description of planar semimodular lattices. $\mathbf{J}(L)$ denotes the order of all nonzero join-irreducible elements of L and $\mathbf{J}_0(K)$ is $\mathbf{J}(L) \cup 0$.

Let X, Y be posets. The *disjoint sum* $X + Y$ of X and Y is the set of all elements in X and Y considered as disjoint. The relation $\leq$ keeps its meaning in X and in Y, while neither $x \geq y$ nor $x \leq y$ for all $x \in X, y \in Y$.

If R is a rectangular slim semimodular lattice then $\mathbf{J}(R)$ is the disjoint sum of two chains C_1 and C_2 wich means that that $x \in C_1, x \in C_2$ are incomparable. The *width* $\mathrm{w}(P)$ of a (finite) order P is defined to be $\max\{n\colon P$ has an n-element antichain$\}$. The width of $\mathbf{J}(L)$ is called the *dimension* of a semimodular lattice L and will be denoted by $\mathbf{dim}(R)$. An other dimension concept is $\mathbf{Dim}(L)$. $n = \mathbf{Dim}(L)$ is the greatest integer such that L contains a sublattice isomorphic to the 2^n-element boolean lattice. If L is a distributive lattice then $\mathbf{dim}(L) = \mathbf{Dim}(L)$. On the other hand $\mathbf{Dim}(M_3) = 2$ and $\mathbf{dim}(M_3) = 3$.

C_n denotes an n-element chain. By G. Czédli, E. T. Schmidt, [2] $\mathbf{dim}(L) = 3$ is equivalent to the condition: there 3 disjoint chains C_n, C_m and C_k, $\mathbf{J}(R) = C_n \cup C_m \cup C_k$, $n \leq m \leq k$ such that R is the cover-preserving join-homomorphism of $G = G_R = C_n \times C_m \times C_k$. In this case we say that R is of type $(n.m, k)$. Two dimensional semimodular lattices are the slim lattices. G_R is called the (lower) *grid* of R.

Remark: the upper grid of a semimodular lattice L is $\overline{G} = C^3$, where C is a chain which has the same length as L. By [2] L is the cover-preserving join-homomorphic image of $\overline{G}$.

Regularity can be defined for arbitrary dimension:

Definition 1. *A* rectangular *lattice L is a finite semimodular lattice in which* $\mathbf{J}(L)$ *is the disjoint sum of chains.*

If you has a slim rectangular lattice R visually, this looks like to Figure 1, i.e. – properly drawn – we see the contour in Figure 2.

If you has a planar semimodular lattice and if you draw properly then you get Figure 2, in the non slim case the dimension is grater then 2, the lattice in Figure 2 has dimesion 5

Let R be a 3-dimensional rectangular semimodular lattice. If we have 3 disjoint chains C_n, C_m and C_k then rectangular means that $\mathbf{J}(R)$ is the disjoint sum of

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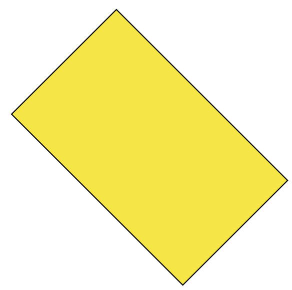

FIGURE 1. A rectangular slim semimodular lattice.

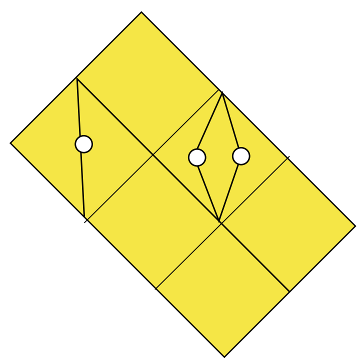

FIGURE 2. A rectangular planar semimodular lattice

these chains. How does it looks like R? The first answer is, visually we see Figure 3, if you draw "properly". The direct product $G = C_n \times C_m \times C_k$ is such a lattice, which looks like to Figure 3. There is an other lattice of this type: this is $M_3[C_n]$, see Figure 5 (here as patchwork of covering squares and M_3-s). It is interesting that this is modular.

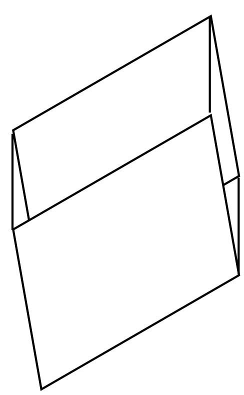

FIGURE 3. The contour of a 3D rectangular semimodular lattice, a cuboid.

The expression "it looks like" is not an exact property, to define this exactly we introduce the following concept:

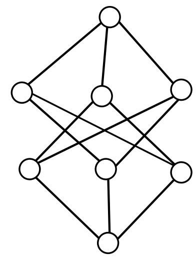

FIGURE 4. Improperly drawn 8-element boolean lattice.

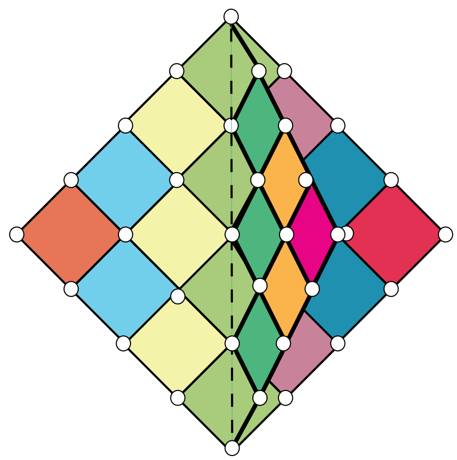

FIGURE 5. $M_3[C_4]$ as patchwork.

Definition 2. *The* skeleton *of a 3D semimodular lattice is an eight-element boolean lattice which contains* 0 *and* 1.

The skeleton of L will be denoted by $\mathbf{Sk}(L)$. The skeleton of a 2D semimodular lattice is a four-element boolean lattice which contains 0 and 1.

That the 3-dimensional lattice R looks like to Figure 3 means that R contains a skeleton, in this case this means that $\mathbf{dim}(R) = \mathbf{Dim}(R)$. It is easy to see that $\mathbf{Sk}(R) = \mathbf{Sk}(G_R) = \mathbf{Sk}(\overline{G}_R)$. In this paper we would like to describe the 3-dimensional rectangular lattices especially those which don't contains a skeleton.

2. EXAMPLES

We consider first, some exemplars. The simplest case is that $\mathbf{J}(L)$ is the 3-element antichain, i.e. L is join-generated by the following order P, see in Figure 6;

Then we have a lattice of type $(1, 1, 1)$. Take the order P in Figure 6 this is a chopped lattice, where $a \wedge b = a \wedge c = b \wedge c = 0$. The semimodular lattices join-generated by P are the eight-element boolean lattice and M_3 the diamond.

If L is of type $(2, 2, 2)$ the we have the order Q, see in Figure 7. The cover-preserving join-homomorphic images of ${C_2}^3$ (with more then 2 elements) are the eight-element boolean lattice and M_3.

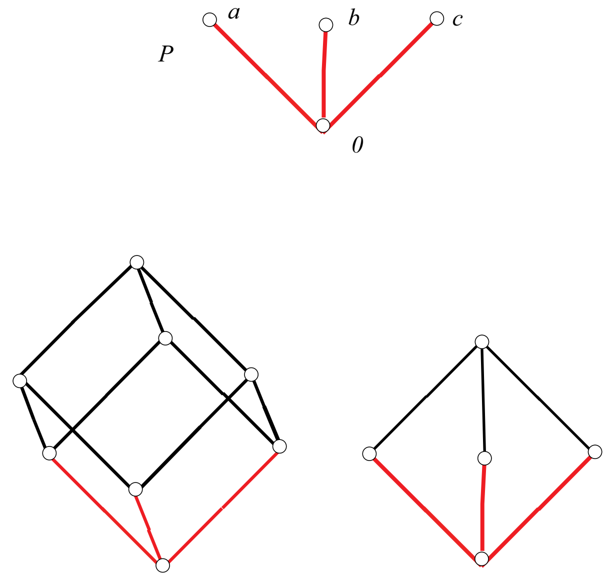

FIGURE 6. The poset P and the join generated semimodular lattices.

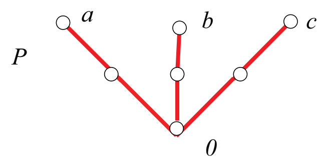

FIGURE 7. The poset Q.

Then Q is a chopped lattice which join generates in the class of semimodular lattices either ${C_3}^3$ this is a **cube** or $M_3[C_3]$.

Remark. This is equivalent to the following: ${C_2}^3$ has only one non trivial cover-preserving join-congruence, which is where the dual atoms with 1 form a congruence class.

These generates either ${C_2}^3$ or $M_3[C_3]$.

$M_3[C_3]$ is a cover-preserving join-homomorphic image of ${C_2}^3$. You can see this lattice in the cover of Algebra Universalis. On the home page of E. T. Schmidt there is a rotary example.

It follows that we can consider $M_3[C_3]$ as the **"modular cube"** .

${C_2}^3$ and $M_3[C_3]$- as geometric shapes have 6–6 flaps.

In Figure 5. we have a 3-dimensional patchwork of 8 monochromatic cubes.

3. THE SOURCE

To describe the cover-preserving join- congruences of ${C_n}^3$ we need the notion of source elements.

Let Θ be a cover-preserving join-congruence of a distributive lattice G (which is not necessarily the grid).

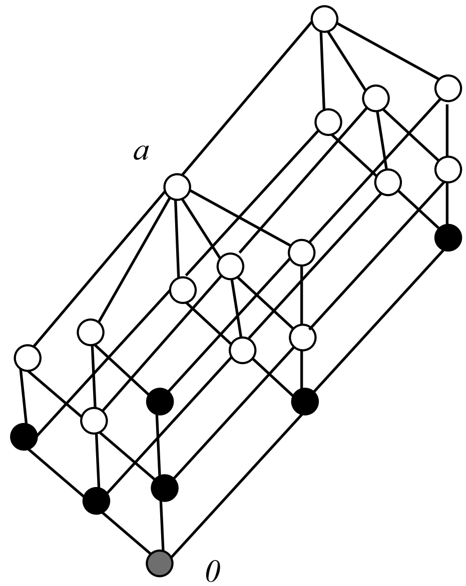

FIGURE 8. A semimodular lattice of type $(2,2,2)$ with $\mathbf{J}_0(K)$.

Definition 3. *An element $s \in G$ is called a* source element *of* Θ *if there is a* $t, t \prec s$ *such that* $s \equiv t\ (\Theta)$ *and for every prime quotient* u/v *if* $s/t \searrow u/v, s \neq u$ *imply* $u \not\equiv v\ (\Theta)$. *The set* $\mathcal{S}_\Theta$ *of all source elements of* Θ *is the* source *of* Θ.

Lemma 1. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s\ (\Theta)$. If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z\ (\Theta)$.*

Proof. Let s be a source element of Θ then $s \equiv t\ (\Theta)$ for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t\ (\Theta)$. This implies $x \wedge t \not\equiv t\ (\Theta)$. By Lemma 1 we have $x \equiv s\ (\Theta)$.

To prove that $v \not\equiv z\ (\Theta)$, we may assume that $v \prec s$. Take t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely $.x, v, t$. These generate an eight-element boolean lattice in which $s \equiv t\ (\Theta)$, $s \equiv x\ (\Theta)$ and $s \equiv v\ (\Theta)$. By the choice of t we know that $v \not\equiv v \wedge t\ (\Theta)$, $x \not\equiv x \wedge t\ (\Theta)$ and $z \not\equiv x \wedge t \wedge v\ (\Theta)$. It follows that $x \not\equiv t\ (\Theta)$, otherwise by the transitivity $x \not\equiv v\ (\Theta)$. This implies $t \wedge x \not\equiv t \wedge x \wedge v\ (\Theta)$. Take the covering square $\{x \wedge v \wedge t, z, t \wedge x, x\}$ then by Lemma 1 $z \not\equiv x\ (\Theta)$, which implies $z \not\equiv v\ (\Theta)$. □

The following results are proved in [4]. The source $\mathcal{S}$ satisfies an independence property:

Definition 4. *Two elements s_1 and s_2 of a distributive lattice are* s-independent *if $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$. A subset S is s-independent iff every pair $\{s_1, s_2\}$ is s-independent.*

$G = C_n \times C_m \times C_k$ can be considered as a 3D hypermatrix, this has a row and two colums. G contains covering cubes, these are called cells. the source elements are top element of the sources., see Figure 9.

Lemma 2. *Every row/column contains at most one source element.*

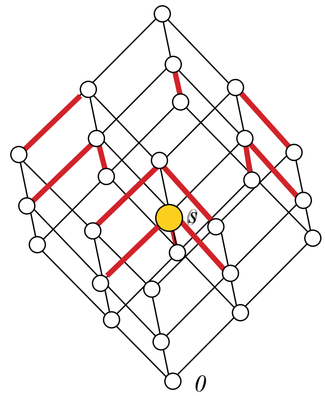

FIGURE 9. The representation of a $(2,2,2)$- type lattice

Lemma 3. *Two elements s_1 and s_2 of a distributive lattice are s-independent if one of the following is satisfied:*

(1) s_1 and s_2 are incomparable,
(2) $s_1 < s_2$ and $t \prec s_2$ implies $t \geq s_1$, i.e. $s_1 \leq s_2{}^*$.

Proof. It is clear that for an incomparable pair s_1, s_2 if $u \prec s_1$ and $v \prec s_2$ then s_1/u and s_2/v cannot be projective. On the other case, if $s_1 < s_2$ then $t||s_1$ would imply that s_2/t and $s_1/t \wedge s_1$ are perspective. This means that $t \geq s_1$. □

It is easy to prove that every s-independent subset $\mathcal{S}$ generate a cover-preserving join-congruence Θ. The semimodular lattice L is characterized by (G, Θ) or $(G, \mathcal{S})$, where $\mathcal{S}$ is an s-independent subset. We write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Theorem. *A rectangular 3D semimodular lattice R of type (n, m, k) has a 3-skeleton if and only if the source S of R has less then n elements.*

Proof. Let R be a rectangular 3D semimodular lattice of type (n, m, k), $(n \leq m \leq k)$.

Then we have the grid $G = C_n \times C_m \times C_k$ a cover-preserving join-congruence Θ and the source S of Θ such that $R \cong G/\Theta$. Then G has a skeleton $\mathbf{Sk}(G) = \{0, a, b, c, p, q, r, 1\}$, see in Figure 10 (in this example the type is of type $(2,2,2)$). Take the ideal generated by $\mathbf{a} = (c_n, 0, 0)$ of G (in Figure 10 the yellow line, this is the leading line). $[(c_{n-1}, 0, 0), (c_n, c_m, c_k)]$ is the first row of the "matrix" G.

If every row contains a source element, i.e. we have n source elements then $p \equiv 1(\Theta)$. This implies that $\mathbf{Sk}(G)/\Theta$ is not a boolean lattice, i.e. R has no skeleton.

Assume that we have rows without source elements. then $p \not\equiv 1(\Theta)$. Then the image of $\mathbf{Sk}(G)$ by the cover-preserving join-homomorphism $\varphi : G \to R$ is the skeleton $\mathbf{Sk}(R)$ of R.

□

Take M_3, then $\mathbf{J}(M_3)$ is the three-element antichain, i.e. $G = {C_2}^3$. G has only one cell, the unit elemet is a source element. Let Θ be the corresponding cover-preserving join-congrurnce. The facor lattice is $G/\Theta = M_3$. If $R = N_7$ then rhe grid is $G = {C_3}^2$ the matrix is not invertible and R and R has a skeleton.

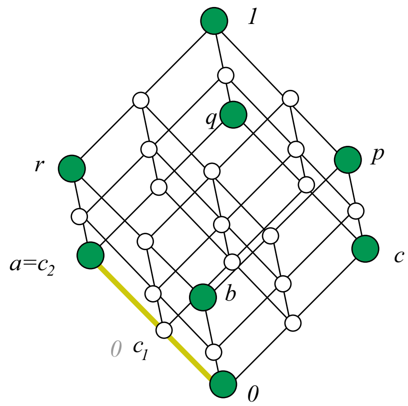

FIGURE 10. The skeleton of a G.

4. SOME MORE EXAMPLES

Example 1.
We would like to describe all lattices of type $(1, 2, 2)$. If R is such a lattice, then the grid is $G = C_2 \times C_3 \times C_3$. This has 4 cells, one row and two columns.

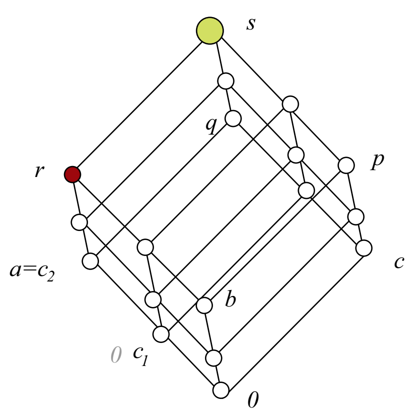

FIGURE 11. The skeleton $G = C_2 \times C_3 \times C_3$.

Example 2.

5. IMPORTANT RECTANGULAR LATTICES: PATCH LATICES

5.1. **Patch lattices.** Let R be a 3D semimodular lattice of type (n, n, n).
R is called a *patch lattice* if in every row and column s except the last row and last (two) columns is a source elements. The patch lattices were introduced in [2]

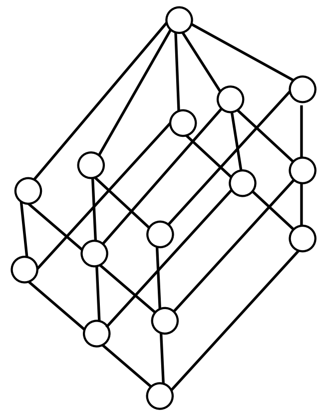

FIGURE 12. S_{15} of type $(1, 2, 2)$

for the two dimensional semimodular lattices, these are the building stones. Every path lattice R has a skeleton $\mathbf{Sk}(R)$. **The dual atoms of $\mathbf{Sk}(R)$ are dual atoms of R.**

5.2. **The matrix representation.** . Take the grid G of R. This can be considered as a hypermatrix. If the cell is labeled by a source element then we write as entry 1 into this cell. otherwise we write 0. then we have a $(0, 1)$-hypermatrix, see Figure 12.

From Theorem 1 it follows that the condition R has no skeleton is equivalent to the condition "the hypermatrix is invertible".

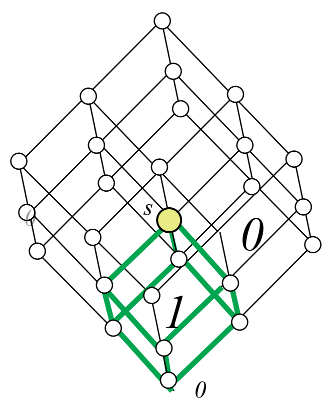

FIGURE 13. A cell labeled by a source element s.

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