Laplace Transformation

1. Basic notions

Definition

For any complex valued function f defined for $t > 0$ and complex number s, one defines the *Laplace transform of* $f(t)$ by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

if the above improper integral converges.

Notation

We use $\mathsf{L}(f(t))$ to denote the Laplace transform of $f(t)$.

Remark

It is clear that Laplace transformation is a linear operation: for any constants a and b:

$$\mathsf{L}(af(t) + bg(t)) = a\mathsf{L}(f(t)) + b\mathsf{L}(g(t)).$$

Remark

It is evident that $F(s)$ may exist for certain values of s only. For instance, if $f(t) = t$, the Laplace transform of $f(t)$ is given by (using integration by parts : $u \mathrel{\hat{=}} t, v' \mathrel{\hat{=}} e^{-st}$):

$$\int te^{-st}\, dt = -\frac{t}{s}e^{-st} - (-\frac{1}{s}\int e^{-st}\, dt) = -\frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st} \rightsquigarrow$$

$$\rightsquigarrow \int_0^\infty te^{-st}\, dt = -\frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2} \text{ if } s > 0 \text{ and does not exists if } s \leq 0\,.$$

Therefore $\quad \mathsf{L}(t) = \dfrac{1}{s^2}\,.$

Theorem

If $f(t)$ is a piecewise continuous function defined for $t \geq 0$ and satisfies the inequality $|f(t)| \leq Me^{pt}$ for all $t \geq 0$ and for some real constants p and M, then the Laplace transform $\mathsf{L}f(t))$ is well defined for all $\operatorname{Re} s > p$.

Illustration

The function $f(t) = e^{3t}$ has Laplace transform defined for any $\operatorname{Re} s > 3$, while $g(t) = \sin kt$ has Laplace transform defined for any $\operatorname{Re} s > 0$. The tables on the following page give the Laplace transforms of some elementary functions.

Remark

It is clear from the definition of Laplace Transform that if $f(t) = g(t)$, for $t \geq 0$, then $F(s) = G(s)$. For instance, if $H(t)$ is *the unit step function* defined in the following way: $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$, then $\boldsymbol{\mathsf{L}(H(t)) = \mathsf{L}(1) = \frac{1}{s}}$ and (as we have seen above) $\boldsymbol{\mathsf{L}(H(t)t) = \mathsf{L}(t) = \frac{1}{s^2}}$. Generally, $\boldsymbol{\mathsf{L}(H(t)t^n) = \mathsf{L}(t^n) = \frac{n\,!}{s^{n+1}}}$.

2. Inverse Laplace Transforms

Definition

If, for a given function $F(s)$, we can find a function $f(t)$ such that $\mathsf{L}(f(t)) = F(s)$, then $f(t)$ is called the *inverse Laplace transform of* $F(s)$. Notation: $f(t) = \mathsf{L}^{-1}(F(s))$.

Examples

$\mathsf{L}^{-1}\left(\frac{1}{s^2}\right) = t\,.\quad \mathsf{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin\omega\, t}{\omega}$ (hiszen $\mathsf{L}(\sin\omega\, t) = \frac{\omega}{s^2+\omega^2}$ és L lineáris.)

We are not going to give you an explicit formula for computing the inverse Laplace Transform of a given function of s. Instead, numerous examples will be given to show how $\mathsf{L}^{-1}(F(s))$ may be evaluated. It turns out that with the aide of a table and some techniques from elementary algebra, we are able to find $\mathsf{L}^{-1}(F(s))$ for a large number of functions.

Our first example illustrates the usefulness of the decomposition to partial fractions:

Example

$$\frac{5s^2+3s+1}{(s^2+1)(s+2)} = \frac{2s-1}{s^2+1} + \frac{3}{s+2} \quad\rightsquigarrow\quad \mathsf{L}^{-1}\left(\frac{5s^2+3s+1}{(s^2+1)(s+2)}\right) = 2\cos t - \sin t + 3e^{-2t}.$$

3. Some simple properties of Laplace Transform

3.1 Transform of derivatives and integrals

If f and f_0 are continuous for $t > 0$ such that $f(t)e^{-st} \longrightarrow 0$ as $t \longrightarrow \infty$, then we may integrate by parts to obtain ($F(s) = \mathsf{L}(f(t))$)

$$(1)\qquad \boldsymbol{\mathsf{L}(f'(t))} = \int_0^\infty e^{-st} f'(t)\,dt = \boldsymbol{sF(s) - f(0)}$$

$$\Big(\text{indeed by } u' = f'(t)\,, v = e^{-st}: \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\big|_0^\infty - s\int_0^\infty f(t)e^{-st}\,dt = -f(0) - sF(s)\Big)$$

and applying this formula again (assuming the apropriate conditions concerning the function and it first and second derivative hold):

$$\mathsf{L}(f''(t)) = s\mathsf{L}(f'(t)) - f'(0) = s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)\,.$$

Similarly (again assuming the apropriate conditions concerning the derivatives hold) we obtain the general formula:

$$\mathsf{L}(f^{(n)}(t)) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

Example

$\mathsf{L}(\cos\, t) = \mathsf{L}(\sin\, 't) = s\mathsf{L}(\sin\, t) - \sin\, 0 = \frac{s}{s^2+1}$

It follows from (1) that

$$(*)\quad \mathsf{L}(f(t)) = F(s) = \frac{1}{s}(\mathsf{L}(f'(t)) + f(0))$$

Example

$$\mathsf{L}(\sin^2\, t) = \frac{1}{s}(\mathsf{L}(\sin\, 2t) + 0) = \frac{2}{s(s^2+4)}$$

Corollary

If f is continuous, then $\mathsf{L}(\int_0^t f(\tau)d\tau) = \frac{1}{s}\mathsf{L}(f(t))$.

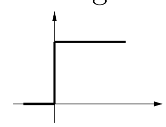
(Indeed (*) can be applied to the function $g(t) = \int_0^t f(\tau)\,d\tau$.)

3.2 Transform of shifts in s and t

(a) If $\mathsf{L}(f(t)) = F(s)$, then $\mathsf{L}(e^{at}f(t)) = F(s-a)$ for any real constant a.

Note that $F(s-a)$ represents a shift of the function $F(s)$ by a units to the right.

(b) The unit step function $s(t) = 0$, ha $t < 0$ és $s(t) = 1$, ha $t \geq 0$:



If $a > 0$ and $\mathsf{L}(f(t)) = F(s)$, then $\mathsf{L}(f(t-a)\cdot s(t-a)) = F(s)e^{-as}$.

Example

Since $s^2 - 2s + 10 = (s-1)^2 + 9$, we have

$$\frac{s+2}{s^2-2s+10} = \frac{s-1}{(s-1)^2+9} + \frac{3}{(s-1)^2+9}\text{, így } \mathsf{L}^{-1}\left(\frac{s+2}{s^2-2s+10}\right) = e^t(\cos 3t + \sin 3t).$$

3.3 Transform of power multipliers

If $\mathsf{L}(f(t)) = F(s)$, then

$$\mathsf{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s)$$

for any positive integer n, particularly $\mathsf{L}(tf(t)) = (-1)F'(s)$.

3.4 Convolution

Definition

Given two functions f and g, we define, for any $t > 0$,

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx.$$

The function $f * g$ is called the *convolution of f and g.*

Remark The convolution is commutative.

Theorem (*The convolution theorem*)

$$\mathsf{L}((f*g)(t)) = \mathsf{L}(f(t))\cdot\mathsf{L}(g(t)).$$

In other words, if $\mathsf{L}(f(t)) = F(s)$ and $\mathsf{L}(g(t)) = G(s)$, then $\mathsf{L}^{-1}(F(s)G(s)) = (f*g)(t)$.

Example

$$\mathsf{L}^{-1}\left(\frac{s}{(s^2+\omega^2)^2}\right)=\mathsf{L}^{-1}\left(\frac{s}{s^2+\omega^2}\cdot\frac{1}{s^2+\omega^2}\right)=\mathsf{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right)*\mathsf{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right)=$$

$$=\cos\,\omega\,t*\frac{\sin\,\omega\,t}{\omega}=\frac{1}{\omega}\int_0^t\cos\,\omega\,x\,\sin\,\omega(t-x)\,dx=$$

$$=\frac{1}{\omega^2}\left(\frac{1}{4}\cos(-2\,\omega\,x+\omega\,t)+\frac{1}{2}\,t\,\omega\,\sin(\omega\,t)\right)\bigg|_0^t=$$

$$=\frac{1}{\omega^2}\left(\frac{1}{4}\cos(\omega\,t)+\frac{1}{2}\,t\,\omega\sin(\omega\,t)\right)-\frac{1}{\omega}\left(\frac{1}{4}\cos(\omega\,t)\right)=\frac{1}{2\,\omega}\,t\sin(\omega\,t)\,,$$

where in order to integrate, we have used addition formulas for the trigonometric functions.

A simpler example: $\mathsf{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)=\mathsf{L}^{-1}\left(\frac{1}{s}\frac{1}{s^2+1}\right)=1*\sin\,t=\int_0^t\sin(t-x)\,dx=$

$$=\int_0^t(\sin t\,\cos x-\cos t\,\sin x)dx=\sin t\sin x\big|_0^t+\cos t\sin x\big|_0^t=\sin^2 t+\cos^2 t-\cos t=$$

$=1-\cos\,t\,.$ Indeed, $\mathsf{L}(1-\cos\,t)=\frac{1}{s}-\frac{s}{s^2+1}=\frac{s^2+1-s^2}{s(s^2+1)}=\frac{1}{s(s^2+1)}\,.$

3.5 Laplace Transform of a periodic function

Definition

A function f is said to be *periodic* if there is a constant $T>0$ such that $f(t+T)=f(t)$ for every t. The constant T is called *the period of* f.

The sine and cosine functions are important examples of periodic function. One other example is the periodic triangular wave. It is is the function defined by $f(t)=t$ if $0\le t\le 1$, $f(t)=2-t$ if $1\le t\le 2$ and $f(t+2)=f(t)$ for any t.

The following proposition is useful in calculating the Laplace Transform of a periodic function.

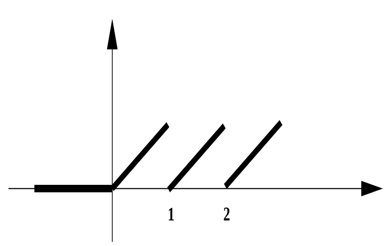
Proposition

Let f be a periodic function with period T and f_1 is one period of the function, Then (as usual $F(s)=\mathsf{L}(f(t))$):

$$F(s)=\frac{\mathsf{L}(f_1(t))}{1-e^{-Ts}}=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt\,.$$

Example

$f(t)=0$ ha $t<0$, $f(t)=t$ ha $0\le t\le 1$ és $f(t+n)=f(t)$ tetszőleges n-re:

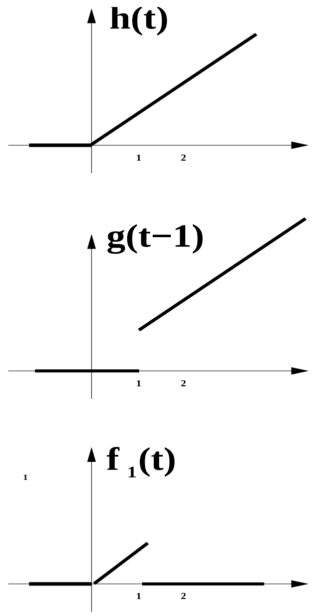


Now $f_1(t) = 0$ if $t < 0$ and $t > 1$, further $f_1(t) = t$ if $0 \leq t < 1$, then defining

$$h(t) = 0 \text{ if } t < 0 \text{ and } h(t) = t \text{ otherwise}$$

$$g(t) = 0 \text{ if } t < 0 \text{ and } g(t) = t + 1 \text{ otherwise,}$$

we have $f_1(t) = h(t) - g(t-1)$:



Therefore, $\mathsf{L}(f_1(t)) = \mathsf{L}(h(t)) - \mathsf{L}(g(t-1)) = \dfrac{1}{s^2} - e^{-s}\left(\dfrac{1}{s^2} + \dfrac{1}{s}\right) = \dfrac{1 - e^{-s} - s\,e^{-s}}{s^2}$,

that is $\mathsf{L}(f(t)) = \dfrac{\mathsf{L}(f_1(t))}{1 - e^{-s}} = \dfrac{1 - e^{-s} - s\,e^{-s}}{s^2(1 - e^{-s})}$.