Stochastic processes

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This lecture is based on
Essentials of Stochastic processes
book of Rick Durrett

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2020 File B

Introduction

In this file we give some proofs and slightly deeper analysis of notions introduced in the first File.

Notation

We study discrete time Markov chain X_n on the countable (finite or countably infinite) state space S with transition matrix $P = (p(i,j))_{i,j \in S}$.

$$\mathbb{P}_x(A) := \mathbb{P}(A | X_0 = x).$$

$\mathbb{E}_x$ notates the expected value for the probability $\mathbb{P}_x$. We frequently use the hitting time:

$$T_y := \min\{n \geq 1 : X_n = y.\}$$

Notation (cont.)

The probability that the chain of starting at x will ever get to y:

$$\rho_{xy} := \mathbb{P}_x\left(T_y < \infty\right)$$

Intuitively: we feel that ρ_{yy}^2 is the probability of the event that {starting from y, we will come back to y twice} because we feel that whatever happens after we got back to y first is independent of what had happened before. To make this feeling precise we introduce the notion of **stopping time** or **Markov-time**.

Notation (cont.)

Definition 1.1 (Stopping time)

T is a *stoppingtime* if we can decide whether the event $\{T = n\}$ (we stop at time n) occur or does not occur by looking at the values $X_0, \dots, X_n$.

Stopping time

We can see easily that T_y is a stopping time, because

$$\{T_y = n\} = \{X_1 \neq y, \dots, X_{n-1} \neq y, X_n = y\}.$$

Example 1.2

- $T \equiv k$ constant time is stopping time.
- The first time when X_n enters a given set A. $T(A) := \min\{n : X_n \in A\}$ is a stopping time.
- For a fixed k: the first time when the process enters into a given $A \subset S$ set for the k^{th} time is also a stopping time. (We will prove this later.)

Stopping time (cont.)

Counter example: The last time when the process enters a given set is not a stopping time because we need to know the whole future to check it.

Lemma 1.3

The

- *sum*
- *maximum*
- *minimum*

of two stopping times is stopping time.

The proof is trivial.

Strong Markov property

Theorem 1.4

Let X_n be Markov chain with transition matrix: $\mathbf{P} = (p(i,j))$ and T be a stopping time. Assuming that $T = n$ and $X_T = y$, every further piece of information about $X_0, \dots, X_T$ is irrelevant for the future (to estimate values of X_{T+k}) and for $k \geq 0$: X_{T+k} behaves like the original Markov chain started from y.

In the case of $T \equiv k$ we get back the Markov property. We only prove now that

$$(1) \qquad \mathbb{P}\left(X_{T+1} = z | X_T = y, T = n\right) = p(y,z).$$

Strong Markov property (cont.)

For an arbitrary $\mathbf{x} = (x_0, \dots, x_n)$, where $x_i \in S$, let $X_0^n(\mathbf{x})$ be and event defined by

$$X_0^n(\mathbf{x}) = \{X_0 = x_0, \dots, X_n = x_n\}.$$

We define

$$V_n := \{\mathbf{x} : X_0^n(\mathbf{x}) \Longrightarrow (T = n \text{ and } X_T = y)\}.$$

In other words: V_n is the set of those $\mathbf{x} = (x_0, \dots, x_n)$, for which:

$$X_0 = x_0, \dots, X_n = x_n \Longrightarrow T = n \text{ and } X_T = y$$

Strong Markov property (cont.)

$$\mathbb{P}\left(X_{T+1}=z, X_T=y, T=n\right) =$$
$$= \sum_{\mathbf{x}\in V_n} \mathbb{P}\left(X_{n+1}=z, X_0^n(\mathbf{x})\right) =$$
$$= \sum_{\mathbf{x}\in V_n} \underbrace{\mathbb{P}\left(X_{n+1}=z \middle| X_0^n(\mathbf{x})\right)}_{p(y,z)} \cdot \mathbb{P}\left(X_0^n(\mathbf{x})\right) =$$
$$= p(y,z) \sum_{\mathbf{x}\in V_n} \mathbb{P}\left(X_0^n(\mathbf{x})\right) =$$
$$= p(y,z)\cdot \mathbb{P}\left(T=n, X_T=y\right).$$

We divide both sides by $\mathbb{P}\left(T=n, X_T=y\right)$ and this yields (1).

Recurrent and transient states

Let $T_y^1 := T_y$ and

$$T_y^k := \min\left\{n > T_y^{k-1} : X_n = y\right\}$$

the time of the k^{th} return to y. Because of the strong Markov property

$$\mathbb{P}_y\left(T_y^k < \infty\right) = \rho_{yy}^k.$$

Recurrent and transient states (cont.)

- If $\rho_{yy} < 1$, then the probability of the event that the chain process comes back to y: $\rho_{yy}^k \to 0$. Thus, there's a time when the process no longer gets back to y. These y states are called transient.
- If $\rho_{yy} = 1$. Then for $\forall k$: $\rho_{yy}^k = 1$. Thus the process gets back to y infinitely many times. Then these y states are called recurrent.

Recurrent and transient states (cont.)

The following simple observation will be useful:

Lemma 2.1

If $\mathbb{P}_x(T_y \leq k) \geq \alpha > 0 \; \forall x \in S$, *then*

$$\mathbb{P}_x(T_y > nk) \leq (1-\alpha)^n.$$

Namely, the probability that in the first n steps we have not visited y is less than $1-\alpha$, the same is true for the subsequent $n-1$ blocks of paths of length k.

Recurrent and transient states (cont.)

Definition 2.2

We say that x **communicates with** y ($x \rightsquigarrow y$) if the probability of reaching y from x in some (not necessarily in one) steps is positive. In other words:

$$x \rightsquigarrow y \text{ if } \rho_{xy} = \mathbb{P}_x \left(T_y < \infty \right) > 0.$$

It follows from Markov property that

$$\text{(2)} \qquad \text{If } x \rightsquigarrow y \text{ and } y \rightsquigarrow z \text{ then } x \rightsquigarrow z.$$

Recurrent and transient states (cont.)

Lemma 2.3

If $\rho_{xy} > 0$ and $\rho_{yx} < 1$, then x is transient.

This is trivial, because since the event {starting from x we can get to y in finitely many steps} has positive probability and the event {from y we don't get back to x} also has positive probability. By Markov property: {starting from x we never get back to x} has also positive probability, so x is transient.

Recurrence and transience

Unless we say otherwise, we do not assume that $\#S < \infty$. Recall:

$$T_y^k = \min\left\{n > T_y^{k-1} : X_n = y\right\}$$

and $\rho_{xy} = \mathbb{P}_x\left(T_y < \infty\right)$.

From the strong Markov property:

$$\mathbb{P}_x\left(T_y^k < \infty\right) = \rho_{xy} \cdot \rho_{yy}^{k-1} \tag{3}$$

Let

$$N(y) := \#\left\{n \geq 1 : X_n = y\right\}.$$

Recurrence and transience (cont.)

Obviously,

$$\{N(y) \geq k\} = \left\{T_y^k < \infty\right\}. \tag{4}$$

Hence, whenever $\rho_{yy} < 1$ (that is y is transient) we have

$$\begin{aligned}
\mathbb{E}_x N(y) &= \sum_{k=1}^{\infty} \mathbb{P}_x \left(N(y) \geq k\right) = \sum_{k=1}^{\infty} \mathbb{P}_x \left\{T_y^k < \infty\right\} \\
&\stackrel{(3)}{=} \rho_{xy} \sum_{k=1}^{\infty} \rho_{yy}^{k-1} = \frac{\rho_{xy}}{1 - \rho_{yy}}
\end{aligned}$$

Recurrence and transience (cont.)

So have obtained that

$$(5) \qquad \rho_{yy} < 1 \Longrightarrow \mathbb{E}_x N(y) = \frac{\rho_{xy}}{1-\rho_{yy}}.$$

That's why $\mathbb{E}_y N(y) < \infty$ iff $\rho_{yy} < 1$. On the other hand we will prove hat

Lemma 2.4

$$\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} p^n(x,y).$$

Recurrence and transience (cont.)

Proof.

$N(y) = \sum_{n=1}^{\infty} \mathbb{1}_{X_n=y}$. Taking expected value:

$$\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} \mathbb{E}_x \left[\mathbb{1}_{X_n=y}\right] = \sum_{n=1}^{\infty} \underbrace{\mathbb{P}_x \left(X_n = y\right)}_{p^n(x,y)} = \sum_{n=1}^{\infty} p^n(x, y).$$

□

As a corollary of Lemma 2.4 and (5) we get:

Recurrence and transience (cont.)

Theorem 2.5

An element $y \in S$ is recurrent if and only if:

$$\sum_{n=1}^{\infty} p^n(y,y) = \mathbb{E}_y\left[N(y)\right] = \infty.$$

Now we prove, using Theorem 2.5 that the **Simple Symmetric Random Walk (SSRW)** on $\mathbb{Z}$ is recurrent. Recall that SSRW is defined on $\mathbb{Z}$ by the transition probability matrix:

$$p(i, i+1) = p(i, i-1) = \frac{1}{2}, \quad \text{for all } i \in \mathbb{Z}.$$

Theorem 2.6

SSRW is null-recurrent on $\mathbb{Z}$. (The same is true on $\mathbb{Z}^2$, but the SSRW is transient in $\mathbb{Z}^d$ for $d \geq 3$.)

You can read more on this topic in [1]. We use Stirling-formula in the proof:

(6) $$1 < \frac{n!}{\sqrt{2\pi n} \cdot (n/e)^n} < e^{1/(12n)}.$$

Hence we get

(7) $$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}},$$

where $\sim$ means that the ratio of the two sides tends to 1.

Proof

First we prove that SSRW is recurrent on $\mathbb{Z}$.

Remark: Starting from 0 we get to 0 in $2n$ steps iff we make n steps to the right and n steps to the left. The probability of each of these paths is $(1/2)^{2n}$ and the number of these paths is $\binom{2n}{n}$.

Proof (Cont.)

Hence,

$$p^{2n}(0,0) = \binom{2n}{n}(1/2)^{2n} \sim \frac{1}{\sqrt{\pi n}},$$

where we used the formula given in (7). So,

$$\sum_{n=1}^{\infty} p^n(0,0) \geq \sum_{n=1}^{\infty} p^{2n}(0,0) = \mathrm{const} \cdot \sum_{n=1}^{\infty} n^{-1/2} = \infty.$$

Now we use Theorem 2.5 to conclude that the simple symmetric random walk on $\mathbb{Z}$ is recurrent.

Proof (Cont.)

Now we prove null-recurrence: Let E_k be the expected number of steps required to reach k starting from 0 for the first time. By definition, E_0 is not zero but the expected number of steps of the first return to 0. If we want to get into $k > 1$ from 0, first we have to reach 1, then 2, and so on; and the expected number of getting from i to $i+1$ is the same for all $i \in \mathbb{Z}$. Hence,

$$E_k = kE_1.$$

Proof (Cont.)

From the 1-step argument:

$$E_1 = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot E_2,$$

because from -1 we can get into 1 in two steps. From this:

$$E_1 = 1 + E_1 \text{ so } \boxed{E_1 = \infty}.$$

Then by the 1-step argumnet we get

$$E_0 = 1 + \frac{1}{2} E_{-1} + \frac{1}{2} E_1,$$

So $\boxed{E_0 = \infty}$, thus the chain is null-recurrent.

SVEN ERICK ALM
Sven Erick Alm
Simple random walk. Click here for the online version.

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