Stochastic processes

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This lecture is based on
Essentials of Stochastic processes
book of Rick Durrett

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2020 File F

1. Normal distribution, Gaussian process
2. Brownian motion

Definition 1.1 (Normal distribution (on $\mathbb{R}$))

Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Random variable X with parameters (n, σ^2) has normal (or Gaussian) distribution $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density function:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

If $\mu = 0$ and $\sigma = 1$, then we get the standard normal distribution $\mathcal{N}(0,1)$. We use the following notation this section:

$$(1) \qquad \varphi(x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}, \qquad \Phi(x) := \int_{-\infty}^{x} \varphi(y) dy.$$

Some properties

$X \sim \mathcal{N}(\mu, \sigma^2)$ and $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, 2$. Then

(a) $\mathbb{E}[X] = \mu, \mathrm{Var}(X) = \sigma^2$.

(b) $F_X(x) = \mathbb{P}(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

(c) $X_1 + X_2 = \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(d) $X \sim \mathcal{N}(0, 1)$, then

(2)

$$\frac{1}{\sqrt{2\pi}} \cdot \left(x^{-1} - x^{-3}\right) \cdot e^{-x^2/2} \leq \mathbb{P}\left(X \geq x\right) \leq \frac{1}{\sqrt{2\pi}} \cdot x^{-1} \cdot e^{-x^2/2}$$

(d) $Y \sim \mathrm{Bin}(n, p)$, $a < b$, then

$$(3) \qquad \lim_{n \to \infty} \mathbb{P}\left(a < \frac{Y - np}{\sqrt{np(1-p)}} < b\right) = \Phi(b) - \Phi(a).$$

Multivariate normal distribution

This is a review of matterial taught in the course Probability 1. Let A be a matrix of $d \times d$, which is symmetric, positive definit and $\mathbf{m} \in \mathbb{R}^d$ be a fixed vector. A random variable $\mathbf{X}$ which takes values in $\mathbb{R}^d$ has multivariate normal or Gaussian distribution, if its density function is of the form:

$$f(\mathbf{x}) = \frac{\sqrt{\det(A)}}{(2\pi)^{d/2}} \cdot \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \cdot A \cdot (\mathbf{x}-\mathbf{m})}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Multivariate normal distribution (cont.)

The meaning of the matrix A: Let $\mathbf{X} = (X_1, \dots, X_d)$.

$$\left(A^{-1}\right)_{ij} = \mathbf{Cov}(X_i, X_j) = \mathbb{E}\left[(X_i - \mathbb{E}\left[X_i\right]) \cdot (X_j - \mathbb{E}\left[X_j\right]\right].$$

Multivariate normal distribution (cont.)

Definition 1.2

Let **X** be as above. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A, and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be the ortonormal basis of $\mathbb{R}^d$ with the appropriate eigenvectors. Let us define diagonal matrix

$$D := \mathrm{diag}(\lambda_1, \ldots, \lambda_d).$$

We form the $d \times d$ orthogonal matrix:
$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_d \end{bmatrix}$ by eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ as column vectors.

Multivariate normal distribution (cont.)

Lemma 1.3

Let $\mathbf{X}$ *be as above. Then*

$$\mathbf{X} = P \cdot D^{-1/2} \cdot (Y_1, \dots, Y_d) + \mathbf{m} \tag{4}$$

where $Y_i = \mathcal{N}(0,1), i = 1, \dots, d$ *and they are all independent.*

See [1, chapters 6 and 7].

1. Normal distribution, Gaussian process

2. Brownian motion

Definition of the one-dimensional Brownian motion

(a) If $0 \leq t_0 < t_1 < \cdots < t_n$ are positive numbers, then

$$B(t_0), B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$$

are independent. (Independent increments.)
(b) If $s, t \geq 0$, then

$$\mathbb{P}\left(B(s+t) - B(s) \in A\right) = \int_A (2\pi t)^{-1/2} \exp\left(-\frac{x^2}{2t}\right) dx$$

(Stationary increments.)
(c) With probability one: $t \to B_t$ is continuous.

Constructioning Brownian motion

Let $\mathcal{D}_0 := \{0, 1\}$ and for some $n \geq 1$:

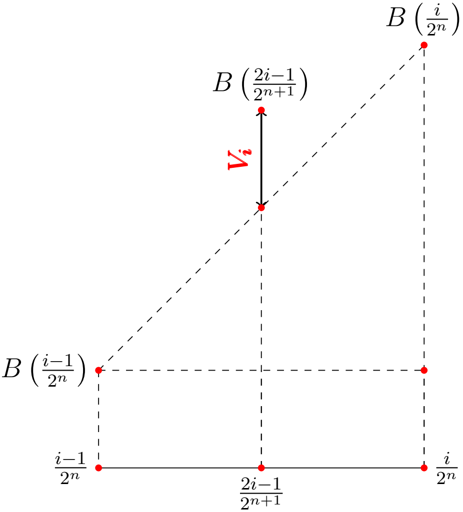
$$\mathcal{D}_n := \left\{ \frac{2k-1}{2^n} : 1 \leq k \leq 2^{n-1}, k \text{ is odd} \right\}$$

$$\mathcal{D} := \bigcup_{n=0}^{\infty} \mathcal{D}_n$$

is the set of dyadic rational numbers in $[0, 1]$.
Construction of random function $B : [0, 1] \to \mathbb{R}$:

Constructioning Brownian motion (cont.)

$$V_i = \mathcal{N}\left(0, \frac{1}{2^{n+2}}\right)$$



Constructioning Brownian motion

- $B(0) := 0,\ B(1) = \mathcal{N}(0,1)$.
- Let us assume that $B\left(\frac{i}{2^n}\right)$, $0 \le i \le 2^n$ have already been defined. Fix a $t \in \mathcal{D}_{n+1}$. Then for some $1 \le i \le 2^n$:

$$t = \frac{2i-1}{2^{n+1}} = \frac{1}{2}\left(\frac{i-1}{2^n} + \frac{i}{2^n}\right),$$

where $B\left(\frac{i-1}{2^n}\right)$ and $B\left(\frac{i}{2^n}\right)$ have already been defined. Let:

(5)

$$B(t) := \mathcal{N}\left(\frac{1}{2}\cdot\left(B\left(\frac{i-1}{2^n}\right)+B\left(\frac{i}{2^n}\right)\right), \frac{1}{2^{n+2}}\right).$$

Constructioning Brownian motion (cont.)

- for $x \in (0,1)$ let

$$B(x) := \lim_{t \uparrow x,\ t \in \mathcal{D}} B(t) = \lim_{t \downarrow x,\ t \in \mathcal{D}} B(t). \tag{6}$$

To see that this definition is correct, we must prove that the limits above exist and if $x \in \mathcal{D}$, then these limits are equal to $B(x)$.

Theorem 2.1

$n \geq 1$ is fixed. For $i = 1, \dots, 2^n$:

$$Z_i := B\left(\frac{i}{2^n}\right) - B\left(\frac{i-1}{2^n}\right). \tag{7}$$

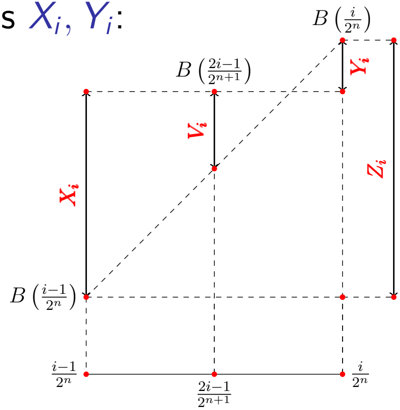
Then

- *(a)* $\{Z_i\}_{i=1}^{2^n}$ *are independent,*
- *(b)* $\forall i$: $Z_i \in \mathcal{N}\left(0, \frac{1}{2^n}\right)$.
- *(c) If* $s, t \in \mathcal{D}$, $s < t$, *then*

$$B(t) - B(s) \in \mathcal{N}(0, s - t). \tag{8}$$

Proof I

Let us assume that for some n the statement is true. We have defined variables V_i and Z_i for $i = 1, \ldots, 2^n$. We use induction to prove the theorem. Let us introduce random variables X_i, Y_i:



Proof II

We know that the following random variables are independent:

$$Z_1, \ldots, Z_{2^n}, V_1, \ldots, V_{2^n}.$$

Their joint density function at $(\mathbf{z}, \mathbf{v}) = (z_1, \ldots, z_{2^n}, v_1, \ldots, v_{2^n})$ is as follows

$$f_{\mathbf{Z},\mathbf{V}}(\mathbf{z}, \mathbf{v}) = \prod_{i=1}^{2^n} \frac{2^{n/2}}{\sqrt{2\pi}} \exp\left(-\frac{z_i^2 2^n}{2}\right) \cdot \frac{2^{(n+2)/2}}{\sqrt{2\pi}} \exp\left(-\frac{v_i^2 2^{n+2}}{2}\right)$$

We have just used the inductive assumption and that $Var(Z_i) = \frac{1}{2^n}$. $\mathrm{Var}(V_i) = 2^{-(n+2)}$ by definition (see (5)).

Proof III

Let $G:\mathbb{R}^{2^n}\times\mathbb{R}^{\cdot 2^n}\to\mathbb{R}^{2^n}\times\mathbb{R}^{\cdot 2^n}$

$$(9)\qquad (\mathbf{x},\mathbf{y})=G(\mathbf{z},\mathbf{v}):=\left(\frac{1}{2}\mathbf{z}+\mathbf{v},\frac{1}{2}\mathbf{z}-\mathbf{v}\right).$$

The same coordinate wise: for $1\leq i\leq 2^n$:

$$(10)\qquad x_i:=\frac{1}{2}z_i+v_i,\quad y_i:=\frac{1}{2}z_i-v_i.$$

Using formula (20) from Appendix:

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=f_{\mathbf{Z},\mathbf{V}}(G^{-1}(\mathbf{x},\mathbf{y}))\cdot\left|\det G'(G^{-1}(\mathbf{x},\mathbf{y}))\right|^{-1},$$

where $(\mathbf{z},\mathbf{v})=G^{-1}(\mathbf{x},\mathbf{y})$; with coordinates:
$z_i=x_i+y_i$ and $v_i=\frac{1}{2}(y_i-x_i)$.

Proof IV

Writing these into the definition of $f_{\mathbf{Z},\mathbf{V}}(\mathbf{z},\mathbf{v})$ we get that the joint density function of random variables

$$(X_1,\dots,X_{2^n},Y_1,\dots,Y_{2^n})$$

is

$$\prod_{i=1}^{2^n}\frac{2^{(n+1)/2}}{\sqrt{2\pi}}\exp\left(-\frac{x_i^2 2^{n+1}}{2}\right)\cdot\frac{2^{(n+1)/2}}{\sqrt{2\pi}}\exp\left(-\frac{y_i^2 2^{n+1}}{2}\right)$$

This means that

$$X_1,\dots,X_{2^n},Y_1,\dots,Y_{2^n}$$

are independent random variables of $\mathcal{N}(0,1/2^{n+1})$. From this, proving part (c) is trivial. ■

Equivalent definition

(i) $B(t)$ is a Gaussian process (all of its finite dimensional distributions are Gaussian (normal)).

(ii) $\mathbb{E}\left[B_s\right] = 0$ and $\mathbb{E}\left[B_s \cdot B_t\right] = s \wedge t := \min\left\{s, t\right\}$.

(iii) With probability one: $t \to B_t$ is continuous.

Basic properties

(a) Let $T, \lambda > 0$. Then

$$\mathbb{P}\left(\sup_{t\in[0,T]} |B_t| \geq \lambda\right) \leq \frac{T}{\lambda^2}.$$

(b) Scaling independence:

$$t \to a^{-1/2}B(at)$$

is also a Brownian motion.
(c)time reversal:

$$t \to tB(1/t)$$

is also a Brownian motion.

Basic properties II

(d) Law of the iterated logarithm:

$$(11) \qquad \limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t \log\log t}} = 1 \quad \text{a.s}$$

(e) $B(t)$ is Hölder for every $\alpha < \frac{1}{2}$ for class α, but not for $\alpha = \frac{1}{2}$.

(f) The trajectories of Brownian motion are nowhere differentiable almost surely.

Mirroring Theorem

Let τ be the time when the Brownian motion starting from zero first reaches the previously fixed number a. Let

$$\widehat{B} := \begin{cases} B(t), & \text{ha } t < \tau; \\ a - (B(t) - a), & \text{ha } t > \tau. \end{cases}$$

We get the graph of it by mirroring graph of $B(t)$ to the horisontal line in interval $t > \tau$. $\widehat{B}(t)$ is also a Brownian motion.

Brownian motion in $\mathbb{R}^d$

Definition 2.2

Brownian motion in $\mathbb{R}^d$ is a random continuous function $\mathbf{B} : [0, \infty) \to \mathbb{R}^d$ with the following properties:

(a) $\mathbf{B}(0) = \mathbf{0}$.

(b) for $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$: $\{\mathbf{B}(t_i) - \mathbf{B}(t_{i-1})\}_{i=1}^{n}$ are independent.

(c) for $0 \leq s < t$: density function of $\mathbf{B}(t) - \mathbf{B}(s)$:

$$(2\pi(t-s))^{-d/2} \cdot \exp\left(-\frac{\|\mathbf{x}\|^2}{2(t-s)}\right). \tag{12}$$

Brownian motion in $\mathbb{R}^d$ (cont.)

Lemma 2.3

Let $B_1(t), \dots, B_d(t)$ *be independent one-dimensional Brownian motions. Then*

$$\mathbf{B}(t) := (B_1(t), \dots, B_d(t)) \tag{13}$$

is a d-dimensional Brownian motion.

Proof: Density function of $\mathbf{B}(t) - \mathbf{B}(s)$ has form (12).
Namely: density function of $B_i(t)$:
$f_i(x_i) = (2\pi(t-s))^{-1/2} \exp\left(-\frac{x_i^2}{2(t-s)}\right)$. For components of B_i are independent, density function of $\mathbf{B}(t) - \mathbf{B}(s)$ is

Brownian motion in $\mathbb{R}^d$ (cont.)

$\prod_{i=1}^{n} f_i(x_i)$, which has form (12). The remaing part of the proof is even simpler.

Hölder continuity

Theorem 2.4

The d-dimensional Brownian motion is Hölder-continuous for every $\alpha < \frac{1}{2}$. In other words, there exist constants $\varepsilon > 0$ and $c = c(d, \alpha)$ so that for all $|h| < \varepsilon$, $t \geq 0$ and $t + h \geq 0$:

$$|\mathbf{X}(t+h) - \mathbf{X}(t)| \leq c \cdot h^{\alpha}.$$

Appendix: Integration by substitution

Let X, Y be metric spaces and $G : X \to Y$ be a mapping. Let us assume that μ is a measure on X. Then let us define the "push forward" measure $G_*\mu$ on set Y:

$$G_*\mu(B) := \mu(G^{-1}(B)).$$

Appendix: Integration by substitution (cont.)

Theorem 2.5

Let us assume that

- U, V *are separable metric spaces,*
- $G : U \to V$ *is Borel-measurable,*
- μ *is a Borel-measure on* U*,*
- $f : V \to [0, \infty)$*,*

Then

$$\int_V f(v) d(G_* \mu)(v) = \int_U (f \circ G)(u) d\mu(u). \tag{14}$$

Appendix: Integration by substitution (cont.)

We use this theorem in the special case when

- $U, V \subset \mathbb{R}^n$,
- $G : U \to V$ is a bijection $(1-1$ and onto) and C^1,
- $f : V \to [0, \infty)$,
- A measure μ is fixed on U about which we only know that

$$G_*\mu = \mathcal{L}\mathrm{eb}_n, \tag{15}$$

Appendix: Integration by substitution (cont.)

that is:

$$\mu(A) = \mu(G^{-1} \circ G(A)) = \mathcal{L}\mathrm{eb}_n(G(A)), \tag{16}$$

because G is bijection.

From now on, $\mathcal{L}\mathrm{eb}_n$ is the n-dimensional Lebesgue-measure and we write dx when we integrate by $\mathcal{L}\mathrm{eb}_n$.

Question: What is $d\mu(u)$?

Appendix: Integration by substitution (cont.)

Answer:

$$
\begin{aligned}
d\mu(u) &= \frac{d\mu(u)}{d\mathcal{L}\mathrm{eb}_n(u)}du = \lim_{r\to 0}\frac{\mu(B(u,r))}{\mathcal{L}\mathrm{eb}_n(B(u,r))}du \\
&= \lim_{r\to 0}\frac{\mathcal{L}\mathrm{eb}_n\left(\left(G(B(u,r))\right)\right)}{\mathcal{L}\mathrm{eb}_n(B(u,r))}du \qquad (17)\\
&= \left|\det\left(G'(u)\right)\right| du,
\end{aligned}
$$

where $B(u,r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ is the ball in $\mathbb{R}^n$ with center u and radius r.

Appendix: Integration by substitution (cont.)

From here, using formulas (14) and (15) we get the formula of Integration by substitution.

$$(18) \qquad \int\limits_{v \in G(U)} f(v)dv = \int\limits_{u \in U} f(G(u)) \cdot \underbrace{\left|\det\left(G'(u)\right)\right| du}_{d\mu(u)}.$$

Appendix: Distribution transformations

Let X, Y be continuous random variables in $\mathbb{R}^d$, for which

- density function of X, Y: f_X, f_Y
- $Y = G(X)$, where $G : \mathbb{R}^n \to \mathbb{R}^n$ C^1 is bijection.

Then for every Borel set $H \subset \mathbb{R}$:

$$\begin{aligned}\int\limits_{x \in H} f_X(x)dx &= \mathbb{P}(X \in H) = \mathbb{P}(Y \in G(H)) \\ &= \int\limits_{y \in G(H)} f_Y(y)dy \\ &= \int\limits_{x \in H} f_Y(G(x)) \cdot \left|\det\left(G'(x)\right)\right| dx,\end{aligned}$$

Appendix: Distribution transformations (cont.)

where in the last step we have used formula (18) of Integration by substitution. Using that this holds for all Borel sets, we get:

$$f_X(x) = f_Y(G(x)) \cdot \left|\det\left(G'(x)\right)\right|. \tag{19}$$

Applying substitution $y = G(x)$ on this, we get that

$$f_Y(y) = f_X(G^{-1}(y)) \cdot \left|\det G'(G^{-1}(y))\right|^{-1}. \tag{20}$$

Definition of Stochastic process

A probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and an index set T are fixed. Let $X_t, t \in T$ be random variables in probability space $(\Omega, \mathcal{A}, \mathbb{P})$. $X_t, t \in T$ is called stochastic process. We can think of $\{X_t\}$ as the path of a particle moving randomly in the state space S. The position of the particle at time t is given by $X_t \in S$. In the terms of this course, a stochastic process is defined, if given its:

- state space
- index set
- finite dimensional distributions.

Definition of Stochastic process (cont.)

These don't completely determine the stochastic process if the parameter is continuous.

Example: Let $B(t)$ the standard Brownian motion on the line and U be an independent uniform r.v. on $[0,1]$. Furthermore:

$$\widehat{B}(t) := \begin{cases} B(t), & t \neq U; \\ 0, & t = U. \end{cases}$$

Then the finite dimensional distributions of $\widehat{B}(t)$ and $B(t)$ are the same but trajectories of $\widehat{B}(t)$ are not continuous.

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