$g(s) = \sum_{n=0}^{\infty} p_n s^n$

6.4 Consider the branching process with offspring distribution given by $\{p_n\}_{n=0}^{\infty}$. We change this process into an irreducible Markov chain by the following modification:

whenever the population dies out, then the next generation has exactly one new individual. That is $\mathbb{P}(X_{n+1} = 1 | X_n = 0) = p(0,1) = 1$. For which $\{p_n\}_{n=0}^{\infty}$ will this chain be null recurrent, recurrent, transient?

Solution $m := g'(1) = \sum_{n=1}^{\infty} n p_n$. Assume that $p_1 \neq 1$. Then $m \le 1 \Rightarrow$ recurrent, $m > 1$ transient.

Question When is it null recurrent? We use here Athereya, Ney: Branching Processes Springer 1972

First we note that (*) $\mathbb{E}[Z_n] = m^n$. Namely, $Z_n = Y_{n-1,1} + \dots + Y_{n-1,Z_{n-1}}$ where $Y_{n-1,k} \stackrel{d}{=} Y$ & Y is the number of offsprings. $\mathbb{E}[Z_n] = \mathbb{E}[Z_{n-1}] \cdot \mathbb{E}[Y] = \mathbb{E}[Z_{n-1}] \cdot m$. By mathematical induction we get that $\mathbb{E}[Z_n] = m^n$.

We assume that $m = g'(1) = 1$. This is the so-called critical case. Then by (*) $\mathbb{E}[Z_n] = 1$.

$$1 = \mathbb{E}[Z_n] = \mathbb{E}[Z_n | Z_n > 0] \cdot \mathbb{P}(Z_n > 0) + \underbrace{\overbrace{\mathbb{E}[Z_n | Z_n = 0]}^{0} \cdot \mathbb{P}(Z_n = 0)}_{0} \Rightarrow \quad (**) \ \boxed{\mathbb{E}[Z_n | Z_n > 0] = \frac{1}{\mathbb{P}(Z_n > 0)}}$$

Theorem 1 (from Athereya, Ney's book p.19) Assume $m = \mathbb{E}[Y] = 1$ & $\sigma^2 := \mathrm{Var}\, Y < \infty$.

Then (***) $\boxed{\mathbb{P}(Z_n > 0) \sim \frac{2}{n\sigma^2}}$ where $a_n \sim b_n$ means $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$.

We know that our chain is recurrent.

Let $N := \sup\{n \in \mathbb{N} : Z_n > 0\}$. If $N < \infty$ then the chain returns to 0 after $N+1$ step. If $\mathbb{E}[N] = \infty$ then the chain null recurrent.

$\{N \geq n\} = \{Z_n > 0\}$.

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(N \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(Z_n > 0) \sim \frac{2}{6^2} \sum_{n=1}^{\infty} \frac{2}{n} = \infty.$$

That is the chain is null recurrent.

So, we have proved that

$\mathbb{E}[Y] = 1$ & $\underset{0}{\underset{\vee}{Var(Y)}} < \infty \Rightarrow$ the chain in null recurrent.

We can use Poisson distribution of the number of events occurring in a fixed interval of time or space if these events are

- independent
- The average rate of events (λ) is constant over the interval.
- The events are discrete and can be counted individually, and they cannot happen at the same time.

1.) In a 400-page book, there are a total of 200 typographical errors (randomly scattered). What is the probability that page 13 contains more than one typo?

We use Poisson distribution $\lambda = \frac{200}{400} = \frac{1}{2}$

Let X be the number of typos on page 13.

$$\mathbb{P}(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!},$$

$$\mathbb{P}(X>1) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) = 1 - e^{-\frac{1}{2}} - e^{-\frac{1}{2}} \cdot \frac{1}{2} = 1 - 0.6065 - 0.3033 = \underline{\underline{0.0902}}$$

2.) How many raisins should there be on average in a cookie so that a randomly selected cookie contains at least one raisin with a probability of at least 0.99

Let λ be the average number of raisins per cookies. Let X be the (random) number of raisins in a cookie. We want $\mathbb{P}(X\geq 1)\geq 0.99$. We know that $\mathbb{P}(X\geq 1) = 1-\mathbb{P}(X=0) = 1-e^{-\lambda}$. That is

$$1-e^{-\lambda}\geq 0.99 \Rightarrow e^{-\lambda}\leq 0.01 \Rightarrow \lambda \geq -\ln(0.01) \Leftrightarrow \boxed{\lambda \geq 4.6052}.$$

$$\lim_{\substack{n\to\infty \\ p\to 0,\ np\to\lambda}} \binom{n}{k}\cdot p^k(1-p)^{n-k} = \frac{\lambda^k}{k!}\cdot e^{-\lambda}.$$

Roughly speaking: We use Poisson distribution instead of the binomial distribution if we ask how many events happens among many, independent events of small probability.

3.) In a forest running competition, 330 participants received one tick each, and 75 participants received two ticks each. Estimate how many people participated in the race.

Let X be the number of ticks in a participant. The distribution of $X \sim Poi(\lambda)$. We do not know λ now but we will compute it. If n is the number of participants then $\mathbb{P}(X=1) \approx \frac{300}{n}$; $\mathbb{P}(X=2) \approx \frac{75}{n}$.

$$\left.\begin{array}{l} \frac{\lambda^1}{1!} \cdot e^{-\lambda} = \mathbb{P}(X=1) \approx \frac{330}{n} \\ \frac{\lambda^2}{2!} \cdot e^{-\lambda} = \mathbb{P}(X=2) \approx \frac{75}{n} \end{array}\right\} \Rightarrow \lambda = \frac{5}{11}, \quad n = \frac{330 \cdot e^{\lambda}}{\lambda} = \frac{330 \cdot 11}{5} \cdot e^{\frac{5}{11}} = 1143.78$$

$$\frac{\lambda}{2} = \frac{75}{330} \Rightarrow \lambda = \frac{15\cancel{0}}{33\cancel{0}} = \frac{5}{11}$$

That is the best guess is that there was 1144 participants.

In Kazmer's fairly big family, an average of 2.5 glasses break every month (30 days). <u>Question</u> What is the probability that in the next 10 days not a single glass will break?

<u>Solution</u> Let Y be the number of broken glasses during 10 consecutive days. Then Y has Poisson distribution since we are dealing with a fixed time interval and the events (broken glasses) are independent with a given rate λ.
This rate λ is the average number of glasses broken during 10 consecutive days which is $\lambda = \frac{2.5}{30} \cdot 10 = 0.8333$. So, $Y \sim Poi(\lambda)$ for $\lambda = 0.8333$.
The probability of no glasses broken $= \mathbb{P}(Y=0) = \frac{\lambda^0 \cdot e^{-\lambda}}{0!} = e^{-\lambda} = e^{-0.8333} = \underline{\underline{0.4346}}$.