Markov Chains II

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This course is based on the book:
Essentials of Stochastic processes
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The history of Branching Processes

In 1873 Francis Galton asked in Educational Times: what is the probability of dying off of a name, a family dying agnatically? Reverend Henry William Watson answered it and they published a paper together in 1874: On the probability of extinction of families. Thus the correspondent MC is called Galton-Watson process. So we only regard the number of sons in various generations, because they carry on the name.

Branching processes

Let's regard a population, in which the 0^{th} generation only consists of one person and in the n^{th} generation one gives birth to k children (who will be counted in the $(n+1)^{st}$ generation) with probability p_k (independently of each other); with $k = 0, 1, 2, \dots$

Let X_n be the number of individuals in the n^{th}generation. The state space is $\mathbb{N} = \{0, 1, 2, \dots\}$. If $Y_1, Y_2, \dots$ are i.i.d. random variables for which $\mathbb{P}(Y_m = k) = p_k$, then the transition matrix is $p(0, 0) = 1$ and $p(i, j) := \mathbb{P}(Y_1 + \dots + Y_i = j)$ if $i > 0$ and $j \geq 0$.

Special case: The number of children has geometric distribution.

$$p_{\ell} := \mathbf{P}\left(\text{number of children} = \boldsymbol{\ell}\right) = q^{\boldsymbol{\ell}} p.$$

Then element (k, l) of the transition matrix:

$$p(k, \ell) = \binom{k + \ell - 1}{\ell} p^n q^k.$$

Random walks on $\mathbb{Z}^d$

Simple symmetric random walk on $S = \mathbb{Z}^d$:

$$(1) \qquad p(x,y) := \begin{cases} \frac{1}{2d}, & \text{ha } \|\mathbf{x}-\mathbf{y}\| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

General random walk on $S = \mathbb{Z}^d$:
$\mathfrak{p} : \mathbb{Z}^d \to [0,1]$; $\sum_{\mathbf{x}\in\mathbb{Z}^d} \mathfrak{p}(\mathbf{x}) = 1$, and the transition matrix $\mathbf{P} = (p(x,y))$:

$$p(x,y) := \mathfrak{p}(\mathbf{x}-\mathbf{y}).$$

Two stage Markov chains

In this example X_{n+1} is dependent of (X_{n-1}, X_n).

Basketball chain

Consider a basketball player who makes a shot with the following probabilities:

1/2, if both of his previous shots are missed

2/3, if he has hit one of his last two shots

3/4, if he has hit both of his last two shots.

So let $X_n = S$ denote the success and $X_n = M$ denote the miss.

Two stage Markov chains (cont.)

The state space is: $\{SS, SM, MS, MM\}$ and the transition matrix is:

	SS	**SM**	**MS**	**MM**
SS	$3/4$	$1/4$	0	0
SM	0	0	$2/3$	$1/3$
MS	$2/3$	$1/3$	0	0
MM	0	0	$1/2$	$1/2$

Explanation: If $(X_{n-1}, X_n) = (S, M)$, then the probability of $(X_n, X_{n+1}) = (M, S)$ is equal to $2/3$.

Stationary distribution for the Basketball chain

Following the rule shown above to compute stationary distribution $\boldsymbol{\pi}$, we subtract 1 from transition matrix $\mathbf{P}$'s diagonal elements and replace the last column with ones.

$$A = \begin{bmatrix} -1/4 & 1/4 & 0 & 1 \\ 0 & -1 & 2/3 & 1 \\ 2/3 & 1/3 & -1 & 1 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Stationary distribution for the Basketball chain (cont.)

Then $A^{-1} = \begin{pmatrix} -\frac{13}{6} & -\frac{5}{16} & \frac{11}{16} & \frac{43}{24} \\ -\frac{1}{6} & -\frac{17}{16} & -\frac{1}{16} & \frac{31}{24} \\ -1 & -\frac{3}{8} & -\frac{3}{8} & \frac{7}{4} \\ \frac{1}{2} & \frac{3}{16} & \frac{3}{16} & \frac{1}{8} \end{pmatrix}$.

Its last row is $\boldsymbol{\pi}$. Hence,

$$\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8}\right).$$

Stationary distribution for the Basketball chain (cont.)

Reminder: the order of components is (SS,SM,MS,MM). (S: success, M: miss.) So, in the long term the ratio of successes is:

$$\pi_{SS} + \pi_{KS} = \pi_1 + \pi_3 = \frac{1}{2} + \frac{3}{16} = \frac{11}{16}.$$

Ehrenfest chain: Stationary distribution

Example 0.1 (π for the Ehrenfest chain)

Recall the definition of the Ehrenfest chain: Consider the Markov Chain with state space $S := \{0, 1, 2, \ldots, n\}$ and

1. It jumps from 0 to 1 and from n to $n-1$ with probability 1.
2. For any $0 < i < n$, it jumps from i to $i-1$ with probability i/n and from i to $i+1$ with probability $1 - \frac{i}{n}$.

Ehrenfest chain: Stationary distribution (cont.)

Now compute the stationary state for this chain. The transition matrix:

$$\mathbf{P} := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & 0 & \frac{n-1}{n} & 0 & \cdots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{n-2}{n} & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{n-1}{n} & 0 & \frac{1}{n} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Ehrenfest chain: Stationary distribution (cont.)

For $\boldsymbol{\pi}^T \cdot \mathbf{P} = \boldsymbol{\pi}^T$, thus using notation $\pi_{-1} := \pi_{n+1} := 0$ we obtain that:

$$\pi_{k-1}\left(1 - \frac{k-1}{n}\right) + \pi_{k+1}\frac{k+1}{n} = \pi_k, \; k = 0, 1, \ldots, n. \tag{2}$$

We introduce the generating function:

$$g(x) = \sum_{k=0}^{n} x^k \pi_k. \tag{3}$$

Ehrenfest chain: Stationary distribution (cont.)

Multiply both sides of (2) by n and x^k, then sum it up for k from 1 to n:

$$\sum_{k=1}^{n}(n-k+1)x^k\pi_{k-1} + \sum_{k=0}^{n-1}\pi_{k+1}(k+1)x^k = n\underbrace{\sum_{k=0}^{n}x^k\pi_k}_{g(x)}.$$

By obvious manipulations of this formula we obtain:

$$(1+x)g'(x) = ng(x).$$

Ehrenfest chain: Stationary distribution (cont.)

After solving this differential equation we get:

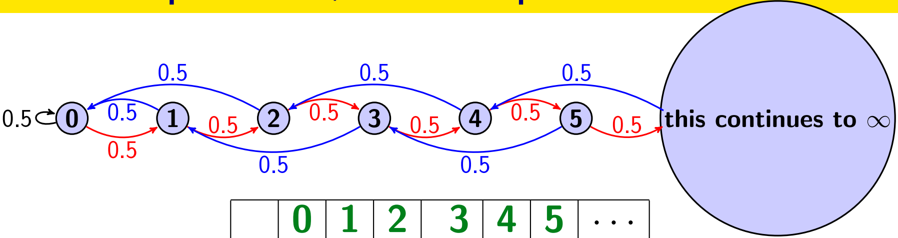
$$g(x) = C(1+x)^n.$$

Using that π is a probability vector we get $g(1) = 1$. Hence $C = 2^{-n}$. That is:

$$g(x) = 2^{-n}(1+x)^n = 2^{-n}\sum_{k=1}^{n}\binom{n}{k}\cdot x^k \tag{4}$$

Compare this to (3) to realize that $\pi_k = 2^{-n}\binom{n}{k}$.

Two steps back, one step forward chain



	0	**1**	**2**	**3**	**4**	**5**	$\cdots$
0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$\cdots$
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\cdots$
2	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$\cdots$
3	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\cdots$
4	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

π for the two steps back one step ahead chain:

From the equation $\boldsymbol{\pi}^T \cdot \mathbf{P} = \boldsymbol{\pi}^T$: $\pi_0 = \frac{1}{2}(\pi_0 + \pi_1 + \pi_2)$ and $\forall k \geq 1 : \pi_k = \frac{1}{2}(\pi_{k-1} + \pi_{k+2})$. From these two equations it comes by induction that

$$\forall k \geq 0 : \pi_k = \pi_{k+1} + \pi_{k+2}. \quad (5)$$

It is obviously satisfied by $\pi_k = (1-\rho)\rho^k$, $k \geq 0$, where ρ is the golden ratio: $\rho = \frac{\sqrt{5}-1}{2}$. Homework: there is no other stationary distribution. So the process spends most of its time (more than 99%) in the set $\{0, 1, \ldots, 9\}$.

Definition 0.2

A MC is doubly stochastic if its probability matrix's column sum equals to 1. $\sum_j p(i,j) = 1$, $\forall j$.

Theorem 0.3

A MC with finite state space is doubly stochastic iff its stationary distribution is the uniform distribution.

Proof.

Let us assume that $\#S = N$, then

$$\sum_x \pi(x) p(x,y) = \frac{1}{N} \sum_x p(x,y) = \frac{1}{N} = \pi(x).$$

Examples

Example 0.4 (Random walk with periodic boundary conditions)

Recall the definition of the random walk with periodic boundary conditions from slide 144 in File MC I. It is obviously doubly stochastic.

Modulo 6 jumps on a circle

Example 0.5

We roll the finite number series $0, 1, 2, \ldots, 5$ on to a circle so that 5 and 0 be neighbours. Then we use such a regular dice which has number

- 1 on three sides,
- 2 on two sides,
- 3 on one side.

We move forward as much as we scored (modulo 6).

Modulo 6 jumps on a circle (cont.)

The transition matrix is:

	0	**1**	**2**	**3**	**4**	**5**
0	0	1/2	1/3	1/6	0	0
1	0	0	1/2	1/3	1/6	0
2	0	0	0	1/2	1/3	1/6
3	1/6	0	0	0	1/2	1/3
4	1/3	1/6	0	0	0	1/2
5	1/2	1/3	1/6	0	0	0

It can easily be seen that the elements of transition matrix's third power $\mathbf{P}^3$ are positive. Thus we see that

Modulo 6 jumps on a circle (cont.)

the chain is irreducible and aperiodic, so the conditions of Convergence Theorem (Theorem 6.2 in File MC I) are satisfied (obviously $\boldsymbol{\pi}(i) = 1/6$, $\forall i$).

Recurrence of the simple Symmetric random walk in $\mathbb{Z}^d$-ben

Theorem 1.1

In $\mathbb{R}^d$ the simple symmetric random walk is recurrent (zero recurrent) if $d = 1$ or $d = 2$ but transient for $d \geq 3$.

We will give the proof in the case when $d = 1$.

One needs to be careful

In the case of countably infinite state space it can happen that there are no recurrent states as the following trivial example shows

Example 1.2 (Monotone increasing MC)

Let S be the set of non-negative integers and $p(i, i+1) := 1$ for all $i \in S$.

Detailed balance condition

$\boldsymbol{\pi}$ satisfies detailed balance condition, if $\forall x, y$

$$\boldsymbol{\pi}(x)p(x,y) = \boldsymbol{\pi}(y)p(y,x) \tag{6}$$

If we sum both sides for y, we get that

$$\sum_y \boldsymbol{\pi}(y)p(y,x) = \boldsymbol{\pi}(x)\underbrace{\sum_y p(x,y)}_{=1} = \boldsymbol{\pi}(x).$$

So, if a probability measure satisfies formula (6), then it is a stationary distribution. There exist stationary

Detailed balance condition (cont.)

distributions which do not satisfy the detailed balance condition (6). For example, consider the MC whose probability matrix is:

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.1 & 0.6 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

Then the stationary distribution π of P does not satisfy (6). To get contradiction, assume that π satisfies (6). From this and from the fact that $p(1,3) = 0$ we get

Detailed balance condition (cont.)

$\pi(3) = 0$. This and formula (6) yield that $\pi(2) = \pi(1) = 0$ which is impossible. On the other hand, $\mathbf{P}$ is a doubly stochastic matrix for which there is a stationary distribution (the uniform distribution): $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

So, it can happen that there is a stationary distribution but it does not satisfy (6). In spite of this, if we have a guess about a probability vector that it could be the stationary distribution, we can check it easily by substituting it into formula (6).

Reversible MC

Now we use [4, chapter 1.6]. **Notation**: For the MC (X_n) we introduce

(7) $$X_0^n := (X_0, \ldots, X_n).$$

So for $\mathbf{x} := (x_0, \ldots, x_n)$

(8) $$\{X_0^n = \mathbf{x}\} = \{X_0 = x_0, \ldots, X_n = x_n\}$$

and for an $\mathbf{x} = (x_0, \ldots, x_n)$ let

(9) $$\overleftarrow{\mathbf{x}} := (x_n, x_{n-1}, \ldots x_1, x_0).$$

Reversible MC (cont.)

It comes easily from formula (6) that:

(10)

$$\pi(x_0)p(x_0, x_1)\cdots p(x_{n-1,}x_n) = \pi(x_n)p(x_n, x_{n-1})\cdots p(x_1, x_0).$$

Using notation $\mathbf{x} = (x_0, \ldots, x_n)$ this implies that:

$$\mathbb{P}_{\pi}\left(X_0^n = \mathbf{x}\right) = \mathbb{P}_{\pi}\left(X_0^n = \overleftarrow{\mathbf{x}}\right). \tag{11}$$

So if MC (X_n) has stationary distribution, **and** it satisfies **detailed balance condition**, then the distribution of

Reversible MC (cont.)

$(X_0, \ldots, X_n)$ is the same as the distribution of $(X_n, \ldots, X_0)$.

Definition 2.1 (reversible MC)

A MC X_n is reversible if it has stationary distribution π and π satisfies the detailed balance condition, that is formula (6) holds.

Reversible MC (cont.)

Example 2.2 (Simple random walk on graphs, slide 137 in File MC I)

Let us regard a simple random walk on graph $G = (V, E)$. Using notation of slide 137 in File MC I, the stationary distribution is: $\pi(y) = \deg(y)/2\#E$. It can be easily seen (homework) that it satisfies detailed balance condition:

$$\pi(x)p(x,y) = \pi(y)p(y,x), \quad \forall x, y \in S.$$

Reversible MC (cont.)

Example 2.3 (Random walk with periodic boundary condition)

Reminder: finite state space (with cardinality N) rolled onto a circle. We jump 1 clockwise with probability p and anticlockwise with probability $q = 1 - p$. The chain is double stochastic, so $\pi = (\frac{1}{N}, \ldots, \frac{1}{N})$. But $\pi(k)p(k, k+1) = \frac{p}{N}$ and $\frac{q}{N} = \pi(k+1)p(k+1, k)$ and they are equal only if $p = q$. So in other instances the detailed balance condition is not satisfied.

Definition 2.4 (Chain with reversed time)

Given an irreducible MC X_n with transition matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$. Let us define the matrix $\widehat{\mathbf{P}} = (\widehat{p}(x,y))$:

$$\widehat{p}(x,y) := \frac{\boldsymbol{\pi}(y)p(y,x)}{\boldsymbol{\pi}(x)}. \tag{12}$$

Then $\widehat{\mathbf{P}}$ is a stochastic matrix (every element is non-negative, the row-sums are 1.) So $\widehat{\mathbf{P}}$ determines a MC $(\widehat{X}_n)$, which we call time reversal of (X_n).

Obviously, if (X_n) is reversible, then $\mathbf{P} = \widehat{\mathbf{P}}$.

Time reversal

Theorem 2.5

Using notation of Definition 2.4:

(a) π *is stationary distribution not only for* (X_n) *but for* $(\widehat{\mathbf{X}}_n)$*, too, and*

(b) *for all* $\mathbf{x}$*:*

$$\mathbf{P}_\pi\left(X_0^n = \mathbf{x}\right) = \mathbf{P}_\pi\left(\widehat{X}_0^n = \overleftarrow{\mathbf{x}}\right), \tag{13}$$

where $\overleftarrow{\mathbf{x}}$ *was defined in* (9).

Time reversal (cont.)

Proof.

Firstly we prove part (a):

$$\sum_y \pi(y)\widehat{p}(y,x) = \sum_y \pi(y)\frac{\pi(x)p(x,y)}{\pi(y)} = \pi(x).$$

Now we see part (b):

$$\begin{aligned}
\mathbb{P}_\pi\left(X_0^n = \mathbf{x}\right) &= \pi(x_0)p(x_0,x_1)p(x_1,x_2)\cdots p(x_{n-1},x_n) \\
&= \pi(x_n)\widehat{p}(x_n,x_{n-1})\cdots\widehat{p}(x_2,x_1)\widehat{p}(x_1,x_0) \\
&= \mathbb{P}_\pi\left(\widehat{X}_0^n = \overleftarrow{\mathbf{x}}\right).
\end{aligned}$$

Birth and death processes

Birth and death processes are those MCs, whose state space are

$$S := \{k, k+1, \dots, n\}.$$

and we cannot jump more than 1. So the possible jumps are: $-1, 0, 1$. The transition probability:

$$p(x, y) = 0 \text{ if } |x - y| > 1 :$$

Birth and death processes (cont.)

Then the transition matrix **P** is:

$$p(x, x+1) = p_x \text{ if } x < n$$
$$p(x, x-1) = q_x \text{ if } x > k$$
$$p(x, x) = 1 - p_x - q_x \text{ if } k \leq x \leq n.$$

and all other $p(x, y) = 0$. Warning: $p + q \neq 1$ is possible here!

Birth and death processes (cont.)

Theorem 2.6

All birth and death processes are reversible.

Birth and death processes (cont.)

Proof

We need to see that we can find a probability measure π on S which satisfies formula (6), thus for $x < n$ it must be true for π:

$$\pi(x+1)\underbrace{p(x+1,x)}_{q_{x+1}} = \pi(x)\underbrace{p(x,x+1)}_{p_x}$$

So, for (6), it is needed that

$$\pi(x+1) = \frac{p_x}{q_{x+1}}\pi(x). \tag{14}$$

Iterating this for every $1 \leq i \leq n-k$

Proof Cont.

$$(15) \qquad \pi(k+i) = \pi(k) \cdot \underbrace{\frac{p_{k+i-1} \cdot p_{k+i-2} \cdots p_{k+1} \cdot p_k}{q_{k+i} \cdot q_{k+i-1} \cdots q_{k+2} \cdot q_{k+1}}}_{r_i}$$

It is easy to see that if we choose $\pi(k)$ such way that

$$(16) \qquad \pi(k) \cdot \left(1 + \sum_{i=1}^{n-k} r_i\right) = 1,$$

then π is a stationary distribution which satisfies the detailed balance condition, so the chain is reversible.

We have computed the stationary distribution for the Ehrenfest Chain (see slide 11). We got that $\pi(k) = 2^{-N}\binom{N}{k}$, but we needed an unpleasant reduction involving generator functions. Now we can easily get this from formula (15) because the Ehrenfest Chain is obviously a birth and death process.

Example 2.7 ($\boldsymbol{\pi}$ for the Ehrenfest chain)

Here: $S = \{0, 1, \ldots, N\}$. From formula (15) we get that

$$r_i = \binom{N}{i} \text{ if } 1 \leq i \leq N.$$

Using that $1 + \sum_{i=1}^{N} r_i = 2^N$ we obtain that for $i = 0, \ldots, N$:

$$\boldsymbol{\pi}(i) = 2^{-N} \binom{N}{i}.$$

Two year collage

Example 3.1 (Two year collage)

At a two year collage the first year students are called freshmen the second year students are the sophomores.

- **Freshmen:** 60% of them become sophomores , 25% of them remain freshmen, 15% of them exit (E) so leave the school.
- **Sophomores:** 70% of them complete the courses with Success (S), 20% of them remain sophomores and 10% of them exit.

Two year collage (cont.)

Then if $S = \{1, 2, G, D\}$ (freshmen, sophomores, Graduate, Drop out) and X_n shows that a student is in which state after n years, then X_n is a MC whose state space is S and its transition matrix:

	1	**2**	**G**	**D**
1	0.25	0.6	0	0.15
2	0	0.2	0.7	0.1
G	0	0	1	0
D	0	0	0	1

Two year collage (cont.)

Let $h(x)$, $x \in S$ be the probability that a student in state x eventually graduates. Then we apply the one step reasoning method. Namely, we do not know $h(1)$ and $h(2)$ but after making one step on the chain the following equations hold:

$$h(1) = 0.25h(1) + 0.6h(2)$$
$$h(2) = 0.2h(2) + 0.7.$$

From this $h(2) = 7/8$ and $h(1) = 0.7$.

Theorem 3.2

A MC is given with a finite state space S*. Let* $a, b \in S$ *and* $C := S \setminus \{a, b\}$*. Let* $h : S \to \mathbb{R}^+$ *be a function satisfying:*

(17)

$$h(a) = 1,\ h(b) = 0, \quad \forall x \in C : \ h(x) = \sum_{y \in S} p(x, y) h(y).$$

Put

$$V_y = \min \{ n \geq 0 : X_n = y \}.$$

Assume that $\forall x \in C$*:* $\mathbb{P}_x (V_a \wedge V_b < \infty) = 1$*. Then*

$$h(x) = \mathbb{P}_x(V_a < V_b).$$

Proof

We frequently use the shorthand notation

$$a \wedge b := \min\{a, b\}.$$

Let $T := V_a \wedge V_b$. By assumption

$$\forall x \in C,\ \mathbb{P}_x(T < \infty) = 1. \tag{18}$$

Proof (cont.)

First we express the probability $\mathbb{P}_x\left(V_a < V_b\right)$ in terms of the expectation of a random variable. Namely, note that by definition,

$$h(X_T) = \begin{cases} 1, & \text{if } V_a < V_b \ ; \\ 0, & \text{if } V_b < V_a. \end{cases}$$

That is

$$h(X_T) = \mathbb{1}_{\{V_a < V_b\}} \tag{19}$$

Proof (cont.)

Hence, for all $x \in C$ we have

$$\mathbb{P}_x\left(V_a < V_b\right) = \mathbb{E}_x\left[h(X_T)\right] \tag{20}$$

Now we prove that

$$\mathbb{E}_x\left[h(X_T)\right] = \lim_{n\to\infty} \mathbb{E}_x\left[h(X_{T\wedge n})\right]. \tag{21}$$

Proof (cont.)

To see this, recall that we assumed that the state space $\#S < \infty$. So, $M := \max_{x \in S} h(x) < \infty$, That is, on the one hand, for all $x \in C$,

$$h(X_{T \wedge n}) < M \text{ holds for all } n. \tag{22}$$

On the other hand, using (18) (which says that T is almost surely finite) we have that

$$\lim_{n \to \infty} h(X_{T \wedge n}) = h(X_T). \tag{23}$$

Proof (cont.)

Putting together (23) and (22), we obtain that (21) holds by Lebesgue Dominated Convergence Theorem. Finally, we verify that

$$(24) \qquad \mathbb{E}_x h(X_{T\wedge n}) = h(x), \ \forall n > 1, \ \forall x \in C.$$

$u_1(x,a) := \mathbb{P}_x\left(X_1 = a\right) = p(x,a)$ and for $k \geq 2$

$$\begin{aligned} u_k &= \mathbb{P}_x\left(X_k = a, T = k\right) \cdot \underbrace{h(a)}_{1} \\ &= \sum_{x_1,\ldots,x_{k-1}\in C} p(x,x_1)p(x_1,x_2)\cdots p(x_{k-1},a). \end{aligned}$$

Proof (cont.)

Moreover, let $S_0 := h(x)$ and

$$S_k := \sum_{x_1,\dots,x_k \in C} p(x,x_1)p(x_1,x_2)\cdots p(x_{k-1},x_k)h(x_k).$$

A careful case analysis yields that by (17) for $k \geq 1$:

$$S_k = S_{k-1} - u_k. \tag{25}$$

Observe that for a $k \leq n$ we have

$$\mathbb{E}_x\left[h(X_{T\wedge n}), T = k\right] = \mathbb{P}_x\left(X_k = a, T = k\right). \tag{26}$$

Proof (cont.)

Using (25), (26), a telescoping sum in the third step and the fact that $S_0 = h(x)$ we obtain:

$$\mathbb{E}_x[h(X_{T\wedge n})] = \mathbb{E}_x[h(X_{T\wedge n}); T > n] + \sum_{k=1}^{n} \mathbb{E}_x[h(X_{T\wedge n}), T = k]$$

$$(27) \quad = S_n + \sum_{k=1}^{n} u_k$$

$$(28) \quad = \underbrace{h(x)}_{S_0} + \sum_{k=1}^{n} (\underbrace{S_k - S_{k-1}}_{-u_k}) + \sum_{k=1}^{n} u_k$$

$$(29) \quad = h(x). \blacksquare$$

Wright-Fisher model

This was introduced on slide 131 in File MC I.
The state space: $S = \{0, 1, \dots, 2N\}$. The absorbing states: 0 and $2N$. Question: what is the probability of ending up in $2N$, or in the model's language: what is the probability that once every gene becomes type a?
The transition matrix:

$$p(x,y) = \underbrace{\binom{2N}{y}\left(\frac{x}{2N}\right)^{y}\left(1-\frac{x}{2N}\right)^{N-y}}_{\text{Binomial}(2N,x/2N)}.$$

Wright-Fisher model (cont.)

That is: the distribution of $y \in \{0, 1, \ldots, 2N\}$ where the Markov chain jumps to from $x \in \{0, 1, \ldots, 2N\}$ is a $\mathrm{Binomial}(2N, x/2N)$ random variable. We know that expected value of a $\mathrm{Binomial}(2N, x/2N)$ r.v. is equal to x. The same in formula:

$$x = \sum_{y=0}^{2N} p(x, y) \cdot y \tag{30}$$

Wright-Fisher model (cont.)

Let us define a function: $h(t) := \frac{t}{2N}$, then by (30):

$$h(x) = \sum_{y=0}^{2N} p(x,y)h(y).$$

Let $a = 2N$ and $b = 0$. Then $h(a) = 1$ and $h(b) = 0$. Obviously:

$$\mathbb{P}_x(V_a \wedge V_b < \infty) > 0, \quad \forall 0 < x < N.$$

Wright-Fisher model (cont.)

So, we can use Theorem 3.2, thus we get:

$$\mathbb{P}_x\left(V_{2N} < V_0\right) = h(x) = \frac{x}{N}. \blacksquare$$

In summary: here we guessed the exit probability function $h(x)$ and to verify our guess we used Theorem 3.2.

Example: Gambler's ruin, unfair case

Now we use the notation introduced on slide 4 in File MC I, where the Gambler's ruin example was introduced with the modification that now $p \neq 1/2$ is arbitrary. Let

$$h(x) = \mathbb{P}_x\left(V_N < V_0\right).$$

That is $h(x)$ is the probability that a gambler starting with $\$x$ eventually wins, that is reaches $\$N$ earlier than $\$0$. Obviously, $h(N) = 1$ and $h(0) = 0$. As usual, let

Example: Gambler's ruin, unfair case (cont.)

$q := 1 - p$ and let $0 < x < N$. Yet again we use the one-step argument: After one step:

$$X_{n+1} = \begin{cases} x+1, & \text{with probability } p; \\ x-1, & \text{with probability } q. \end{cases}$$

So, for $0 < x < N$:

$$h(x) = ph(x+1) + qh(x-1). \tag{31}$$

Example: Gambler's ruin, unfair case (cont.)

Obvious manipulations yield:

$$p\left(h(x+1)-h(x)\right)=q\left(h(x)-h(x-1)\right).$$

Hence,

$$h(x+1)-h(x)=\frac{q}{p}\left(h(x)-h(x-1)\right) \tag{32}$$

Example: Gambler's ruin, unfair case (cont.)

Let $c := h(1) - h(0)$. So, from formula (32) for $x \geq 1$

$$h(x) - h(x-1) = c \left(\frac{q}{p}\right)^{x-1}. \tag{33}$$

Using that $h(N) = 1,\ h(0) = 0$ and a telescopic sum in the second step and (33) in the last step:

$$1 = h(N) - h(0) = \sum_{x=1}^{N} h(x) - h(x-1) = c \sum_{x=1}^{N} \left(\frac{q}{p}\right)^{x-1}.$$

Example: Gambler's ruin, unfair case (cont.)

Put $\theta = q/p$. Then $c = (1-\theta)/(1-\theta^N)$. So

$$h(x) = h(x) - h(0) = c \sum_{i=0}^{x-1} \theta^i = \frac{1-\theta^x}{1-\theta^N}. \tag{34}$$

From here if $N \to \infty$ we get that

$$p > \tfrac{1}{2} \Rightarrow \mathbb{P}_x\left(V_0 = \infty\right) = 1 - \left(\frac{q}{p}\right)^x. \tag{35}$$

Example: Gambler's ruin, unfair case (cont.)

Corollary 3.3

Consider a random walk on $\mathbb{Z}$, in which starting from all $x > 0$ we go forward one step with probability $p > \frac{1}{2}$ and we go backward one step with probability $q = 1 - p$. Then the probability that starting from an arbitrary $x > 0$ we never reach 0 is $1 - \left(\frac{q}{p}\right)^x > 0$. That is every state is transient.

Example: Gambler's ruin, fair case

We consider the Gambler's ruin example with $p = 1/2$.
We use the unfair case ($p \neq 1/2$)'s notation. The argument is the same until formula (32).
But in case of $p = 1/2$ formula (32) shows that the gradient of function $h(x)$ is constant and $h(0) = 0$, $h(N) = 1$ so if $p = 1/2$

$$\mathbb{P}_x (V_N < V_0) = h(x) = x/N.$$

Tennis

The following problem is from [1, p.44].

Tennis (cont.)

Example 3.4

In tennis a player wins the game if either she gets 4 points when the other player has not more than 2 points. If the score is $4-3$ then the winner is the player who makes a two pints advantage first. Assume that

- The server wins the point with 0.6 probability,
- Successive points are independent.

Question: What is the probability that the server wins if the score now is $3-3$?

Tennis (cont.)

Solution: Let X_n be the difference of the points scored from the point of the server after $3-3$ until one of the player has a 2 point advantage so that the game ends. That is the state space is $S := \{-2, -1, 0, 1, 2\}$. Then the transition matrix:

	2	**1**	**0**	**-1**	**-2**
2	1	0	0	0	0
1	0.6	0	**0.4**	0	0
0	0	**0.6**	0	**0.4**	0
-1	0	0	**0.6**	0	0.4
-2	0	0	0	0	1

Tennis

Let $h(x)$ be the probability that the server wins when staring from $X_0 = x$. Obviously now the absorbing states are $\{-2, 2\}$ and $C = \{-1, 0, 1\}$. Clearly,

$$h(2) = 1 \text{ and } h(-2) = 0.$$

From the one-step reasoning:

$$h(x) = \sum_y p(x, y)h(y), \quad \forall x \in C. \tag{36}$$

Tennis (cont.)

$$
\begin{aligned}
(37) \qquad h(1) &= 0.6 \cdot \underbrace{h(2)}_{1} + 0.4h(0) = 0.4h(0) + 0.6 \\
h(0) &= 0.6h(1) + 0.4h(-1) \\
h(-1) &= 0.6h(0) + 0.4 \cdot \underbrace{h(-2)}_{0} = 0.6h(0).
\end{aligned}
$$

Let $\mathbf{R} = (r(x,y))_{x,y \in C}$ be the restriction of matrix $\mathbf{P}$ to rows and columns of C, and let $\widehat{\mathbf{h}}$ be the vector which

Tennis (cont.)

we get by ignoring those coordinates of **h** which are outside C. Then formula (37):

$$\widehat{\mathbf{h}} - \mathbf{R} \cdot \widehat{\mathbf{h}} = \begin{bmatrix} 0.6 \\ 0 \\ 0 \end{bmatrix} \quad (38)$$

Which is:

$$\underbrace{\begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -0.4 \\ 0 & -0.6 & 0 \end{bmatrix}}_{I-R} \cdot \begin{bmatrix} h(1) \\ h(0) \\ h(-1) \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \\ 0 \end{bmatrix}$$

Tennis (cont.)

So

$$\begin{bmatrix} h(1) \\ h(0) \\ h(-1) \end{bmatrix} = (I - R)^{-1} \cdot \begin{bmatrix} 0.6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8769 \\ 0.6923 \\ 0.4154 \end{bmatrix}$$

Exit time from the two year collage

Consider the two-year collage example on slide 46. There,we asked what was the probability of a $k = 1, 2$-year-student of graduating ever. Now, for the same example we ask:

Question: On average, how much time is needed for a student to get out of the school either by completing it successfully or drop out (unsuccessfully).

Let $g(x)$ be the expected number of years that an $x \in \{1, 2\}$-year student leaves the school either because

Exit time from the two year collage (cont.)

she graduates or because she drops out. We define $g(G) = g(D) = 0$. Again, we use the one-step reasoning:

$$\begin{aligned} g(1) &= 1 + 0.25g(1) + 0.6g(2) \\ g(2) &= 1 + 0.2g(2). \end{aligned}$$

This yields: $g(2) = 1.25$ and $g(1) = 2.333$.

Exit time

Theorem 3.5

Let X_n *be a MC with a finite state space* S*. Let* $A \subset S$ *and* $C := S \setminus A$*, and* $V_A := \min\{n \geq 0 : X_n \in A\}$*. Let* $g : S \to \mathbb{R}^+$ *be a function which satisfies:*

(a) $\mathbb{P}_x(V_A < \infty) > 0$, $\forall x \in C$,

(b) $g(a) = 0$, $\forall a \in A$,

Exit time (Cont.)

Theorem 3.5 (Cont.)

(c) $\forall x \in C$

$$g(x) = 1 + \sum_y p(x,y) g(y). \tag{39}$$

Then this function g is the expected exit time. That is

$$g(x) = \mathbb{E}_x \left[V_A \right]. \tag{40}$$

Proof.

The proof goes similarly as the proof of Theorem 3.2. □

Waiting time for TT

Example 3.6

We flip a fair coin until we get two Tails (TT) in a row. Question: what is the expected value of the number of flips?

Solution: We call T the Tails and H the Heads. Let T_{TT} be the (random) number of flips until we get the two Tails (the TT). Now we associate a MC (X_n) with state space $S := \{0, 1, 2\}$, where X_n is the number of consecutive Tails after the n^{th} flip. So, if the n^{th} flip

Waiting time for TT (cont.)

results in a Head, then $X_n = 0$, if it is a Tail, then $X_n = 1$ or $X_n = 2$ depending on X_{n-1} (if it was Head or Tail). State 2 is absorbing because we only flip the coin until this happens. So, the transition matrix:

	0	**1**	**2**
0	1/2	1/2	0
1	1/2	0	1/2
2	0	0	1

Waiting time for TT (cont.)

Let

$$V_2 := \min\{n \geq 0 : X_n = 2\} \text{ and } g(x) := \mathbb{E}_x[V_2].$$

Then from the one-step reasoning:

$$\begin{aligned} g(0) &= 1 + 0.5g(0) + 0.5g(1) \\ g(1) &= 1 + 0.5g(0). \end{aligned} \tag{41}$$

Waiting time for TT (cont.)

Let $\mathbf{1}$ be the vector in $\mathbb{R}^2$, having both components equal to 1. Then $g(0) = 0$ by formula (41):

$$(I - R) \cdot \widehat{\mathbf{g}} = \mathbf{1}, \tag{42}$$

where, as before, R is the matrix we get from **P** by deleting the rows and columns corresponding to the absorbing states (now the only absorbing state is 2) and $\widehat{\mathbf{g}}$ is the vector we get from vector $\mathbf{g}$ by deleting the

Waiting time for TT (cont.)

components belonging to the absorbing states which is 2 as mentioned before. Hence from (42) we get

$$\widehat{\mathbf{g}} = \begin{pmatrix} g(0) \\ g(1) \end{pmatrix} = (I - R)^{-1} \cdot \mathbf{1} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \cdot \mathbf{1} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

So, by Theorem 3.5, we have

$$\mathbb{E}_0 [V_2] = g(0) = \widehat{g}(0) = 6.$$

Tennis at $3-3$

Consider the Tennis problem on slide 68 again.

Question: How long the game lasts if now the score is $4-3$, $3-3$ and $3-4$ from the point of the server?

Solution: Let $g(x)$ be the expected time of the game if $x \in \{1,0,-1\}$. As we discussed, the absorbing states are $A := \{-2,2\}$ and the state space is $S := \{-2,-1,0,1,2\}$. So, $C : A \setminus A = \{1,0,-1\}$.

Using notation analogue to the previous problem:

$$\mathbf{R} = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix}$$

Tennis at $3-3$ (cont.)

and from here:

$$I-R=\begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -0.4 \\ 0 & -0.6 & 1 \end{bmatrix}$$

So, like the previous problem:

$$\begin{pmatrix} g(1) \\ g(0) \\ g(-1) \end{pmatrix} = (I-R)^{-1}\mathbf{1} = \begin{bmatrix} 19/13 & 10/13 & 4/13 \\ 15/13 & 25/13 & 10/13 \\ 9/13 & 15/13 & 19/13 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Tennis at $3-3$ (cont.)

So, at $3-3$ the expected play-time:

$$g(0) = \frac{15+25+10}{13} = 3.846. \tag{43}$$

Tennis at $3-3$ (cont.)

Remark 3.7

Consider an absorbing MC with state space S, absorbing states A and transient states $C := S \setminus A$. Let $y \in C$ and we denote the total number of visit to y including the time 0 if we started from y by $N(y)$. The $N(y) = \sum_{n=0}^{\infty} \mathbb{1}_{X_n=y}$. In this way

Tennis at $3-3$ (cont.)

Remark 3.7 (Cont.)

$$\mathbb{E}_x\left[N(y)\right] = \sum_{n=0}^{\infty} R^n(x,y) = (I-R)^{-1}(x,y). \tag{44}$$

Let T be the duration until the chain gets into an absorbing state. This is equal to the total time the MC spends at all of the transient states together. That is

$$T = \sum_{y \in C} N(y).$$

Hence by (44)

Tennis at $3-3$ (cont.)

Remark 3.7 (Cont.)

$$(45)\ \mathbb{E}_x[T] \; = \; \sum_{y \in C} \mathbb{E}_x[N(y)] = \sum_{y \in C} (I-R)^{-1}(x,y),$$

which is the x-th component of the vector

$$(I-R)^{-1} \cdot \mathbf{1}.$$

With this argument we proved that $(I-R)^{-1}(x,y)$ is equal to the expectation of the number of visits to y (counting the initial state if $x=y$) starting from x.

Tennis at $3-3$ (cont.)

As a Corollary of this Remark we can see that in (43) the summands

$$\frac{15}{13}, \quad \frac{25}{13}, \quad \frac{10}{13}$$

are the expected number of cases when the score is $1, 0, -1$ respectively, before the game ends.

Gambler's ruin, $p = 1/2$: How long does it last?

So: $p(i, i+1) = p(i, i-1) = 1/2$. $A := \{0, N\}$,

$$V_A := \min\{n \geq 0 : X_n \in A\}.$$

Let $g(x) := \mathbb{E}_x [V_A]$. Obviously

$$g(0) = g(N) = 0. \qquad (46)$$

Gambler's ruin, $p = 1/2$: How long does it last? (cont.)

If $0 < x < N$:

$$g(x) = 1 + \frac{1}{2}g(x+1) + \frac{1}{2}g(x-1)$$

$$g(x+1) - g(x) = g(x) - g(x-1) - 2.$$

If $c = g(1) = g(1) - g(0)$, then

$$g(k) - g(k-1) = c - 2(k-1) \tag{47}$$

Gambler's ruin, $p = 1/2$: How long does it last? (cont.)

Using that $g(N) = 0$ and summing the previous equations for $1 \leq k \leq N$, we get telescopic sums. From these:

$$\begin{aligned} 0 &= g(N) = \sum_{k=1}^{N} (g(k) - g(k-1)) \\ &= \sum_{k=1}^{N} (c - 2(k-1)) = cN - 2\frac{N(N-1)}{2}. \end{aligned}$$

Gambler's ruin, $p = 1/2$: How long does it last? (cont.)

Hence, $c = N - 1$. Substituting this back to formula (47) and summing it up we obtain that:

$$g(x) = x(N - x).$$

Gambler's ruin, $p \neq 1/2$: How long does it last?

So, in this case: $p(i, i+1) = p \neq 1/2$ and $p(i, i-1) = 1 - p =: q$. Let $A := \{0, N\}$, $C := \{1, \ldots, N-1\}$,

$$V_A := \min\{n \geq 0 : X_n \in A\}.$$

Let $g(x) := \mathbb{E}_x[V_A]$. Obviously

$$g(0) = g(N) = 0. \tag{48}$$

Gambler's ruin, $p \neq 1/2$: How long does it last? (cont.)

From the one-step reasoning:

$$g(x) = 1 + p \cdot g(x+1) + q \cdot g(x-1), \quad x \in C. \tag{49}$$

These are $N-1$ equations for the $N-1$ unknowns: $(g(1), \ldots, g(N-1))$. This system of equation is the same that appeared in formula (39). Thus, from Theorem 3.5 its solution can only be $g(x) = \mathbb{E}_x\left[V_A\right]$.

Gambler's ruin, $p \neq 1/2$: How long does it last? (cont.)

We can easily check that $g(1), \ldots, g(N-1)$ is the solution of the system of equation (49) if

$$g(x) = \frac{x}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^x}{1-(q/p)^N}, \quad 0 < x < N-1.$$

So, the expected time of the game for $0 < x < N-1$:

$$\mathbb{E}_x\left[V_A\right] = \frac{x}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^x}{1-(q/p)^N}. \tag{50}$$

Gambler's ruin, $p \neq 1/2$: How long does it last? (cont.)

From now we always assume that $x \in C$. We use (50) and distinguish two cases: if $p < q$, then

$$\lim_{N\to\infty} \frac{N}{1-(q/p)^N} = 0 \text{ thus } g(x) \approx \frac{x}{q-p}. \tag{51}$$

On the other hand, if $p > q$, then $(q/p)^N \to 0$, thus

$$g(x) \approx \frac{N-x}{p-q}\left[1-(q/p)^x\right] + \frac{x}{p-q}(q/p)^x. \tag{52}$$

This Subsection is based on Charles M. Grinstead, J. Laurie Snell's book. [2]. Click here for the book.

In this Section (unless we say otherwise) X_n is supposed to be an absorbing MC on a finite state space S with

- transition matrix P,
- absorbing states $A \subset S$ and
- transient states $C := S \setminus A$.

We write $a := \#A$ and $c := \#C$.

We will answer the following questions in general terms:

Questions answered on this Subsection in general terms

(Q1) What is the probability that the process will end up in a given absorbing state? (Theorem 3.11.)

(Q2) What is expected exit time (expectation of the time to get to any of the absorbing states)? (Theorem 3.9.)

(Q3) What is the expected number of visits to a transient state before finally getting to an absorbing state. (Theorem 3.8.)

We always assume that the $c + a$ states of S are arranged as follows: the first c states are the transient states and the last a states are the absorbing states. Then the transition matrix P is in the canonical form:

(53)

	C	**A**
C	$\mathbf{R}$	$\mathbf{Q}$
A	$\mathbf{0}_{a,c}$	$\mathbf{I}_a$

, that is $P = \begin{pmatrix} \mathbf{R} & \mathbf{Q} \\ \mathbf{0}_{a,c} & \mathbf{I}_a \end{pmatrix}$.

where

- R is a $c \times c$ matrix,
- Q is a non-zero $c \times a$ matrix
- $\mathbf{0}_{a,c}$ is an $a \times c$ zero matrix (all elements are zero),
- $\mathbf{I}_a$ is an $a \times a$ identity matrix,

The powers of P

Clearly,

(54) $$P^n = \begin{pmatrix} \mathbf{R}^n & \star \\ \mathbf{0}_{a,c} & \mathbf{I}_a \end{pmatrix},$$

where $\star$ is a $c \times a$ matrix. We have actually proved that

(55) $$\lim_{n\to\infty} \mathbf{R}^n = \mathbf{0}_{c,c}$$

The following Theorem answers question Q3. In special cases we have already seen its proof. Alternatively, for the proof see [2, p. 418, Theorem 11.4].

The fundamental matrix

Theorem 3.8

As always in this Subsection, we assume that X_n is an absorbing MC. Then

- *(a)* $\mathbf{I}_c - \mathbf{R}$ *has an inverse* $\mathbf{N} := (\mathbf{I}_c - \mathbf{R})^{-1}$ *which is called the* ***fundamental matrix***.
- *(b)* $\mathbf{N} = \mathbf{I}_c + \mathbf{R} + \mathbf{R}^2 + \mathbf{R}^3 + \cdots$.
- *(c)* $\mathbf{N} = (n_{i,j})_{i,j=1}^{c}$ *then* $n_{i,j}$ *is the expected values of the times the chain starting from* $i \in C$ *visits* $j \in C$ *before the absorbtion happens. Initial state is counted if* $i = j$.

Time to absorption

Let X_n be as in Theorem 3.8. We write

$$V_A := \min \{ n \geq 0 : X_n \in A \} .$$

We define the vector $\mathbf{g} = (g(x))_{x \in C}$, where

$$g(x) := \mathbb{E}_x [V_A] . \quad \text{where } x \in C$$

That is the $x \in C$-th component $g(x)$ of the vector $\mathbf{g}$ is the expected number of steps until the absorbtion happens if the MC starts from x.

Time to absorption (cont.)

Theorem 3.9

Let X_n be an absorbing MC. We denote the column vector with all components equal to 1 *by* $\mathbf{1} \in \mathbb{R}^c$. *Then*

$$\mathbf{g} = \mathbf{N} \cdot \mathbf{1}. \tag{56}$$

We have actually proved this in the previous subsection in special cases. For a proof see [2, p. 420, Theorem 11.5]. This theorem answers Question Q2. In the following slides we will answer Question Q1.

An auxiliary lemma

We often need the following simple lemma.

Lemma 3.10

Let X be a non-negative integer valued r.v.. Then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k). \tag{57}$$

Proof.

Observe that $X = \sum_{k=1}^{\infty} \mathbb{1}_{\{X \geq k\}}$. Then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{X \geq k\}}\right] = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k).$$ □

Absorption probabilities

Let $\mathbf{B} = (b_{i,j})_{i \in C, j \in A}$ be a $c \times a$ matrix whose elements are defined as follows: for an $i \in C$ and $j \in A$ we write

$$b_{i,j} := \mathbb{P}\,(\text{the chain starting from } i \text{ is absorbed at } j)$$

Theorem 3.11

Let X_n be an absorbing MC. Then

$$\mathbf{B} = \mathbf{N} \cdot \mathbf{Q}. \tag{58}$$

Now we present the proof in a shorter form the we repeat in a more detailed form.

Proof in short

Proof of Theorem 3.11 in short.

Let $\mathbf{R}^0 := \mathbf{I}$. Then

$$\begin{aligned} b_{i,j} &\overset{(57)}{=} \sum_{n=0}^{\infty} \sum_{k \in C} r_{i,k}^{(n)} \cdot q_{k,j} \\ &= \sum_{k \in C} \sum_{n=0}^{\infty} r_{i,k}^{(n)} \cdot q_{k,j} \\ &= \sum_{k \in C} n_{i,k} \cdot q_{k,j} \\ &= (\mathbf{N} \cdot \mathbf{R})_{i,j}. \end{aligned}$$

□

Proof of Theorem 3.11 with details

Proof of Theorem 3.11 with details

Fix an arbitrary $i \in C$ and $j \in A$. Imagine that we start from i and finally arrive at j on such a such path which stay within C before arriving at j. Let m be the length of this path. Observe that $m = 2$ means no states in between i and j on the path and for $m > 2$ there are $n - 2$ states in between i and j on the path and all of them must be in C. So such a path is describe with $c_1, \ldots, c_{m-2} \in C$.

Proof of Theorem 3.11 with details (cont.)

Proof of Theorem 3.11 with details (cont.)

Let us call the probability that such a path is realized $w_{i,c_1,\ldots,c_{m-2},j}$, where the word $c_1, \ldots, c_{m-2}$ is the empty word if $m = 2$. Below we write $n = m - 1$ from the two but last step:

Proof of Theorem 3.11 with details (cont.)

Proof (cont.)

$$
\begin{aligned}
b_{i,j} &= \sum_{m=2}^{\infty} \sum_{c_1,\ldots,c_{m-2}\in C} w_{i,c_1,\ldots,c_{m-2},j} \\
&= \sum_{m=2}^{\infty} \sum_{c_1,\ldots,c_{m-2}\in C} p_{i,c_1} \cdot \prod_{k=1}^{m-2} p_{i_k,c_{k+1}} \cdot p_{c_{n-1},j} \cdot p_{c_{m-1},j} \\
&= \sum_{m=2}^{\infty} \sum_{c_1,\ldots,c_{m-2}\in C} r_{i,c_1} \cdot \prod_{k=1}^{m-2} r_{c_k,c_{k+1}} \cdot q_{c_{m-1},j} \\
&= \sum_{n=0}^{\infty} \sum_{k\in C} r_{i,k}^{(n)} \cdot q_{k,j} \\
&= \Big(\sum_{n=0}^{\infty} \mathbf{R}^n \cdot \mathbf{Q}\Big)_{i,j},
\end{aligned}
$$

Proof of Theorem 3.11 with details (cont.)

Proof (cont.)

where $\prod_{k=1}^{m-2} r_{c_k,c_{k+1}} := 1$ if $m = 2$. Hence

(59) $$B = \sum_{n=0}^{\infty} \mathbf{R}^n \cdot \mathbf{Q}$$

Recall that according to part (b) of Theorem 3.8 we have $\mathbf{N} = \sum_{n=0}^{\infty} \mathbf{R}^n$. Hence, by (59) we obtained that

(60) $$\mathbf{B} = \mathbf{N} \cdot \mathbf{Q}. \qquad \blacksquare$$

The problems considered

In this subsection X_n is an irreducible chain on the finite state space S with transition matrix P and w assume that $\#S \geq 3$. Let $i, j, k \in S$ be three distinct elements of S. We pose the following questions:

(Q4) What is the probability that the chain staring from $i \in S$ visits $j \in S$ earlier than $k \in S$?

(Q5) What is the probability that the chain staring from $j \in S$ returns to j earlier than it visits $k \in S$.

The answer to question Q4

We prepare an absorbing MC from X_n by declaring some of the states absorbing. Namely, let $\mathbf{e}_j$ and $\mathbf{e}_k$ be the coordinate unit vectors in $\mathbb{R}^{\#S}$ which contains a 1 in their j and k-th position respectively, and all other components are zero. We replace of the j-th and k-th rows of P by $\mathbf{e}_j$ and $\mathbf{e}_k$ respectively. The transition probability matrix obtained in this way is denoted by $P^{(j,k)}$ and the corresponding MC is denoted by $X_n^{(j,k)}$.

The answer to question Q4 (cont.)

Clearly, $X_n^{(j,k)}$ is an absorbing MC with absorbing states $A := \{j, k\}$ transient states $C := S \setminus C$. Let

$$P^{(j,k)} = \begin{pmatrix} \mathbf{R}^{(j,k)} & \mathbf{Q}^{(j,k)} \\ \mathbf{0}_{a,c} & \mathbf{I}_a \end{pmatrix}$$

be the canonical form of $P^{(j,k)}$ and let $\mathbf{N}^{(j,k)}$ be the corresponding fundamental matrix:

$$\mathbf{N}^{(j,k)} = \left(I - \mathbf{R}^{(j,k)}\right)^{-1},$$

The answer to question Q4 (cont.)

where I is the $(\#S-2)\times(\#S-2)$ identity matrix.
Now we apply Theorem 3.11 for the MC $X_n^{(j,k)}$. That is we define the $(\#S-2)\times 2$ matrix

$$\mathbf{B}^{(j,k)}=\mathbf{N}^{(j,k)}\cdot\mathbf{Q}^{(j,k)}, \tag{61}$$

where the rows are indexed by the elements of C and the columns are indexed by $\{j,k\}$.

Now we can answer question Q4:

We introduce:

The answer to question Q4 (cont.)

(62)
$\eta_{i,j,k} := \mathbb{P}$(the chain starting from i visists j earlier than k)

Then by Theorem 3.11 and by the definition of matrix B we obtain that

(63) $$\eta_{i,j,k} = \mathbf{b}_{i,j}^{(j,k)},$$

The answer to question Q4 (cont.)

where $\mathbf{b}_{i,j}^{(j,k)}$ is the j-the element of the i-th row of the matrix $\mathbf{B}_{i,j}^{(j,k)}$ defined in (61) and this answers question Q4.

The answer to question Q5

Fix an arbitrary distinct $j, k \in S$. Let $\tau_{j,k}$ be the probability that the chain staring from $j \in S$ returns to j earlier than it visits $k \in S$. We can use the one-step reasoning. Namely, if the chain starting from j returns to j for the first time before visiting k then the chain starting from j cannot make its first step to k. So, in the first step the chain either remains in j (with probability $p(j,j)$ and then it has arrived back to j without visiting k) or it jumps to an $i \neq\in \{j,k\}$ and then it will continue starting now from $i \notin \{j,k\}$ and visits j earlier than k.

The answer to question Q5 (cont.)

The probability of this is (by definition) $\eta_{i,j,k}$. So, the one-step reasoning yields:

$$\tau_{j,k} = p_{j,j} + \sum_{i \notin \{j,k\}} p(j,i) \cdot \eta_{i,j,k} . \tag{64}$$

Example 3.12 (Exercise 1.13 from Lawler's book [5])

Let X_n be a MC on $S = \{1, 2, 3, 4, 5\}$ with

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(a) Is this chain irreducible? Is it aperiodic? **(b)** Find π. **(c)** What is the expected number of steps to return to 1 for the first time if the chain starts from 1? **(d)** What is the expected number of steps to get to 4 for the first time, if the chain starts from 1? **(e)** What is the probability that the chain visits 5 earlier than 3 if the chain starts from 1? **(f)**What is the probability that the chain starting from 3 returns to 3 earlier than it visits 5?

In[71]:=

```
Clear[𝒫, p]
```

$$p = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

```
𝒫 = DiscreteMarkovProcess[1, p]
```

In[74]:=

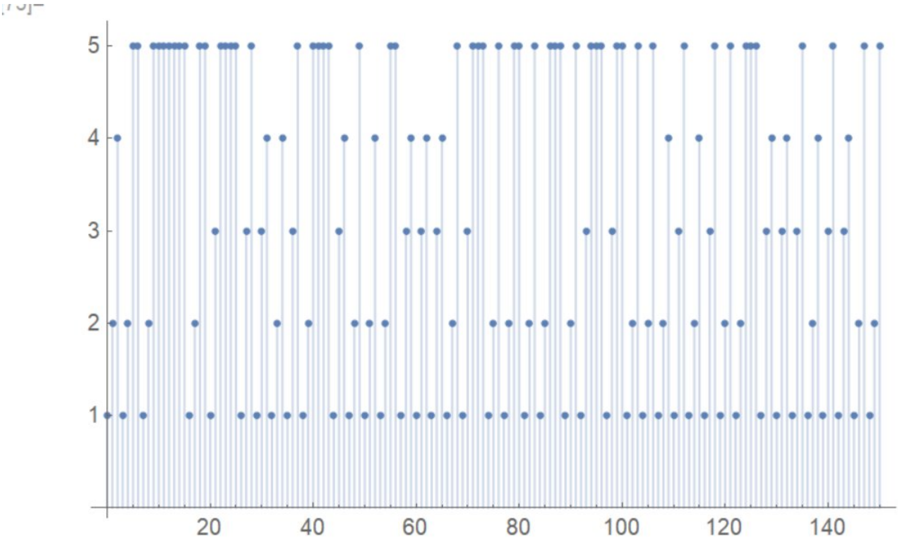
```
data = RandomFunction[𝒫, {0, 150}]
```

Out[74]=

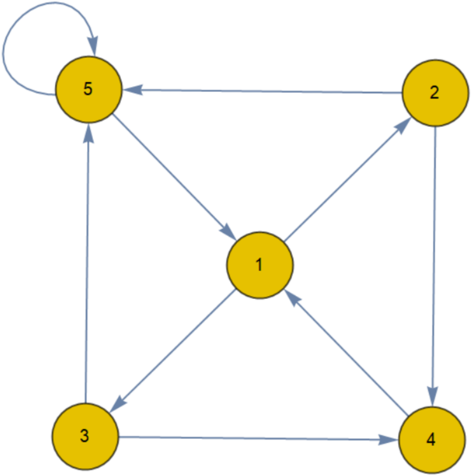
```
TemporalData[ Time: 0 to 150  Data points: 151  Paths: 1 ]
```

In[75]:=

```
ListPlot[data, Filling → Axis, Ticks → {Automatic, {1, 2, 3, 4, 5}}]
```


5
4
3
2
1
20
40
60
80
100
120
140

Graph[𝒫]



Out[102]=

Basic Properties	
InitialProbabilities	
TransitionMatrix	
HoldingTimeMean	{0, 0, 0, 0, 1}
HoldingTimeVariance	{0, 0, 0, 0, 2}
Structural Properties	
CommunicatingClasses	{1, 2, 3, 4, 5}
RecurrentClasses	{1, 2, 3, 4, 5}
TransientClasses	None
AbsorbingClasses	None
PeriodicClasses	None
Periods	{}
Irreducible	True
Primitive	True
Aperiodic	True
Limiting Properties	

The chain is irreducible and aperiodic. This answers (a)

```
invmatrep =
 Inverse[ReplacePart[p - IdentityMatrix[Length[p[[1]]]], {i_, Length[p[[1]]]} :→ 1]]
```

```
invmatrep[[Length[p[[1]]]]]
```

$$\left\{\frac{10}{37}, \frac{5}{37}, \frac{5}{37}, \frac{3}{37}, \frac{14}{37}\right\}$$

```
f[i_, j_] := Mean[FirstPassageTimeDistribution[DiscreteMarkovProcess[i, p], j]]
```

```
Array[f, {5, 5}] // MatrixForm
```

atrixForm=

$$\begin{pmatrix} \frac{37}{10} & \frac{23}{5} & \frac{24}{5} & \frac{34}{3} & \frac{23}{7} \\ \frac{14}{5} & \frac{37}{5} & \frac{38}{5} & \frac{35}{3} & \frac{13}{7} \\ \frac{13}{5} & \frac{36}{5} & \frac{37}{5} & 9 & \frac{19}{7} \\ 1 & \frac{28}{5} & \frac{29}{5} & \frac{37}{3} & \frac{30}{7} \\ 2 & \frac{33}{5} & \frac{34}{5} & \frac{40}{3} & \frac{37}{14} \end{pmatrix}$$

So, **(b)**: $\pi = (\frac{10}{37}, \frac{5}{37}, \frac{5}{37}, \frac{3}{37}, \frac{14}{37})$. **(c)**: 37/10. **(d)**: $\frac{34}{3}$.

Out[92]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

```
In[94]:= r = p[[{1, 2, 4}, {1, 2, 4}]]

In[95]:= q = p[[{1, 2, 4}, {3, 5}]]

In[96]:= id = IdentityMatrix[3]

In[97]:= n = Inverse[id - r]

In[99]:= b = n.q // MatrixForm
```

Out[99]//MatrixForm=

$$\begin{pmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{8}{9} \\ \frac{5}{9} & \frac{4}{9} \end{pmatrix}$$

(e) The answer in 4/9. (The element in the first row since we start from 1 and the column which corresponds to 5 (this is a the second column).

$B^{3,5} =$

	3	5
1	$\frac{5}{9}$	$\frac{4}{9}$
2	$\frac{1}{9}$	$\frac{8}{9}$
4	$\frac{5}{9}$	$\frac{4}{9}$

That is

$$\eta_{1,3,5} = \frac{5}{9}, \quad \eta_{2,3,5} = \frac{1}{9}, \quad \eta_{4,3,5} = \frac{5}{9}. \tag{65}$$

Now we can answer question **(f)** that is we compute $\tau_{3,5}$ which was defined as the probability that the chain staring from 3 returns to 3 earlier than it visits 5. Namely, by (64) we have

$$\begin{aligned} \tau_{3,5} &= p_{3,3} + p(3,1)\eta_{1,3,5} + p(3,2)\eta_{2,3,5} + p(3,4)\eta_{4,3,5} \\ &= 0 + 0 \cdot \frac{5}{9} + 0 \cdot \frac{1}{9} + \frac{2}{5} \cdot \frac{5}{9} \\ &= \frac{2}{9}. \end{aligned}$$

Umbrellas example [1, Excercise 1.37]

Example 3.13

An individual has three umbrellas, some at her office, and some at home. If she is leaving home in the morning (or leaving work at night) and it is raining, she will take an umbrella, if one is there. Otherwise, she gets wet. Assume that independent of the past, it rains on each trip with probability 0.2.

Question 1: Which percentage of time does she get wet?

Umbrellas example (cont.)

We approach this problem in the language of Markov chains. The only idea:
Let $S := \{0, 1, 2, 3\}$ and we write X_n for the number of umbrellas at the current location.
Then the transition matrix P is:

	0	**1**	**2**	**3**
0	0	0	0	1
1	0	0	0.8	0.2
2	0	0.8	0.2	0
3	0.8	0.2	0	0

Umbrellas example (cont.)

In[57]:=
```
Clear[𝒫, p]
```

In[58]:=

$$p = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{8}{10} & \frac{2}{10} \\ 0 & \frac{8}{10} & \frac{2}{10} & 0 \\ \frac{8}{10} & \frac{2}{10} & 0 & 0 \end{pmatrix}$$

In[59]:=
```
invmatrep =
  Inverse[ReplacePart[p - IdentityMatrix[Length[p[[1]]]],
    {i_, Length[p[[1]]]} :> 1]]
invmatrep[[Length[p[[1]]]]]
```

Out[60]=

$$\left\{\frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19}\right\}$$

Umbrellas example (cont.)

This yields that the stationary distribution is

(66) $$\pi = \left(\frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19}\right).$$

Hence it happens with probability $4/19$ that the individual does not have any umbrellas at her current location. However, she does not necessarily get wet at all of these occasions, since there is a rain only every 5th days (independently of everything). So, she gets wet with probability $4/(19 \cdot 5) = 0.04210526\ldots$. Remark: the stationary distribution could be computed by hands easily since the system of equations is very simple.

Umbrellas example (cont.)

Namely, we want to find a probability vector $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ such that
(67)

$$(\pi_0, \pi_1, \pi_2, \pi_3) \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{8}{10} & \frac{2}{10} \\ 0 & \frac{8}{10} & \frac{2}{10} & 0 \\ \frac{8}{10} & \frac{2}{10} & 0 & 0 \end{pmatrix} = (\pi_0, \pi_1, \pi_2, \pi_3)$$

This yields the system of equations:

Umbrellas: answer of Question 1

$$\begin{aligned} 0.8\pi(3) &= \pi(0) \\ 0.8\pi(2) + 0.2\pi(3) &= \pi(1) \\ 0.8\pi(1) + 0.2\pi(2) &= \pi(2) \\ \pi(0) + \pi(1) + \pi(2) + \pi(3) &= 1 \end{aligned} \tag{68}$$

As on slide 37 in File MC I, we throw away the last equation and substituted it by the condition that the sum of the components of π is equal to one, since the last equation of the original system would give no more information than the retained first three equations do. The solution of the system (68) is really obvious high school mathematics.

Notation used in this Section

- In this Section we always that X is a such r.v. which takes only non-negative integers.
- $\forall k \in \mathbb{N}$-re let $p_k := \mathbb{P}(X = k)$.
- The generator function of the r.v. X is

$$g_X(s) := \mathbb{E}\left[s^X\right] = \sum_{k=0}^{\infty} p_k \cdot s^k.$$

The most basic properties of generator functions (in short: g.f.)

Generator functions

(a) A generator function uniquely determines the cumulative distribution function.

(b) The generator function of the sum of two independent r.v. which take only non-negative integers, is the product of the generator functions of these r.v..

Generator functions (cont.)

(c) Let $g(x)$ be the generator function of the r.v. X. Then

$$\mathbb{E}\left[X(X-1)\cdots(X-k)\right] = g^{(k+1)}(1),$$

where $g^{(k+1)}$ is the $k+1$-th derivative of g. Hence by a simple calculation we get:

(69)
$$\mathbb{E}\left[X\right] = g'(1) \text{ és } \mathbb{E}\left[X^2\right] = g''(1) + g'(1).$$

Generator functions (cont.)

(d) $g(1) = 1$ since (p_k) is a probability vector.

Generator functions (cont.)

Lemma 4.1

Let X and N be independent non-negative integer valued r.v. with generator functions g_X és g_N. Moreover, let $X_1, X_2, \dots$ be i.i.d. r.v. having the same distribution as X. We define the r.v.:

$$R := X_1 + \cdots + X_N.$$

Then the generator function of R is:

$$g_R(s) = g_N(g_X(s)). \tag{70}$$

Before the proof of the Lemma we remark that an important corollary of Lemma 4.1 is as follows: Using

Branching Processes with more details

We introduced Branching Processes on slide 3. Given a probability vector $(p_k)_{k=0}^{\infty}$ which we call offspring distribution. A population develops according to the following rule: At the beginning there is one individual on level 0. Then for all $n \geq 0$, each individual on level n independently gives birth to k offsprings with probability p_k. The same with notations:
Let Y be a non-negative integer valued r.v. such that $\mathbb{P}(Y = k) = p_k$. Fix an arbitrary $n \geq 0$. Let X_n denote the number of level n individuals. The level n individuals

Branching Processes with more details (cont.)

$\{1, 2, \ldots, X_n\}$ give birth to $Y_1^{(n)}, \cdots, Y_{X_n}^{(n)}$ individuals. So, the number of level $n+1$ individuals is:

$$X_{n+1} = Y_1^{(n)} + \cdots + Y_{X_n}^{(n)}. \tag{72}$$

We always assume that $\left\{Y_m^{(n)}\right\}_{m,n}$ are i.i.d. r.v. with

$$Y_m^{(n)} \stackrel{d}{=} Y.$$

Branching Processes with more details (cont.)

That is

$$\mathbb{P}\left(Y_m^{(n)} = k\right) = p_k.$$

We can consider (X_n) as a Markov Chain with state space $S = \{0, 1, 2, \dots\}$ and the transition matrix $P = (p_{i,j})$ is given by

(73)
$$p(i,j) = P\left(Y_1 + \cdots + Y_i = j\right) \qquad \text{for } i > 0 \text{ and } j \geq 0,$$

where $\{Y_i\}_{i=1}^{\infty}$ are i.i.d. with $Y_k \stackrel{d}{=} Y$.

Branching Processes with more details (cont.)

Let

$$g_n := \mathbb{E}\left[s^{X_n}\right],$$

That is g_n is the generator function of X_n, (which was defined as the number of level n individuals). Let

$$g(s) := g_1(s) := g_Y(s) = \sum_{n=0}^{\infty} p_n \cdot s^n.$$

Branching Processes with more details (cont.)

Clearly, for all m, the generator function of Y_m is the same:

$$g(s) = g_{Y_m}(s) \quad \forall m.$$

To get a better understanding of the generator function g_n we apply Lemma 4.1 with the following substitutions:

$$X_n \to N, Y_i \to X_i, X_{n+1} \to R.$$

Branching Processes with more details (cont.)

The we obtain from Lemma 4.1 that

$$g_{n+1} = g_n(g(s)).$$

From here, we obtain by mathematical induction that

$$g_n(s) = \underbrace{g \circ \cdots \circ g}_{n}(s) =: g^n(s). \tag{74}$$

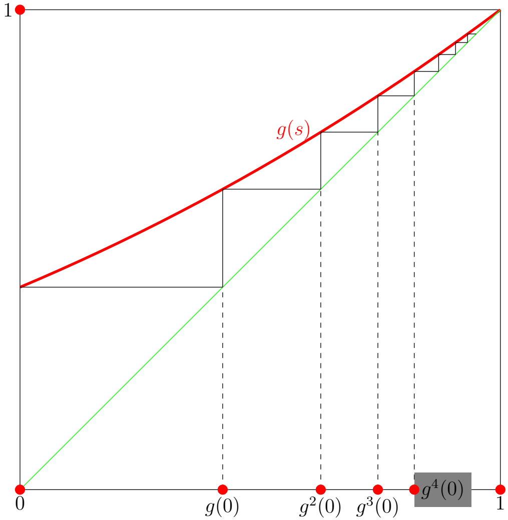
Branching Processes with more details (cont.)

Apply this for $s = 0$ to get:

(75) $$\mathbb{P}(X_n = 0) = g^n(0).$$

Hence $\mathbb{P}(\text{Extinction}) = \lim_{n\to\infty} \mathbb{P}(X_n = 0)$, where Extinction is the event the the Brancing Process dies out in finitely many steps.

$\mathbb{E}[Y]=g'(1)<1 \implies \lim_{n\to\infty} \mathbb{P}(X_n = 0) = 1$



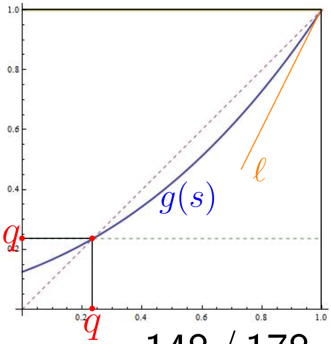
Branching Processes

Summary: p_n is the probability that an individual has exactly n offsprings. Then the expected number of offsprings of an individual is $m := \sum_{n=1}^{\infty} p_n \cdot n$. Consider the generator function:

$$g(s) := \sum_{n=0}^{\infty} p_n \cdot s^n$$

. The graph of g goes through $(1,1)$. Let ℓ be the tangent line to g at $s = 1$. The slope of ℓ is $g'(1) = m$. If $m > 1$ then $\ell \cap [0,1]^2$ is below the line $y = x$. Hence $\exists$ a $q \in [0,1)$ with $g(q) = q$. Looking at the Figure:

$0 \le g'(q) < 1$. So, for $g^n := \underbrace{g \circ \cdots \circ g}_{n}$, we have $g^n(0) \to q$. That is by (75) q is the probability of extinction.



Notation

We study discrete time Markov chain X_n on the countable (finite or countably infinite) state space S with transition matrix $P = (p(i,j))_{i,j \in S}$.

$$\mathbb{P}_x(A) := \mathbb{P}\left(A | X_0 = x\right).$$

$\mathbb{E}_x$ notates the expected value for the probability $\mathbb{P}_x$.
We frequently use the hitting time:

$$T_y := \min\left\{n \geq 1 : X_n = y.\right\}$$

Notation (cont.)

The probability that the chain of starting at x will ever get to y:

$$\rho_{xy} := \mathbb{P}_x\left(T_y < \infty\right)$$

Intuitively: we feel that ρ_{yy}^2 is the probability of the event that {starting from y, we will come back to y twice} because we feel that whatever happens after we got back to y first is independent of what had happened before. To make this feeling precise we introduce the notion of **stopping time** or **Markov-time**.

Notation (cont.)

Definition 5.1 (Stopping time)

T is a *stoppingtime* if we can decide whether the event $\{T = n\}$ (we stop at time n) occur or does not occur by looking at the values $X_0, \ldots, X_n$.

Stopping time

We can see easily that T_y is a stopping time, because

$$\{T_y = n\} = \{X_1 \neq y, \ldots, X_{n-1} \neq y, X_n = y\}\,.$$

Stopping time (cont.)

Example 5.2

- $T \equiv k$ constant time is stopping time.
- The first time when X_n enters a given set A. $T(A) := \min\{n : X_n \in A\}$ is a stopping time.
- For a fixed k: the first time when the process enters into a given $A \subset S$ set for the k^{th} time is also a stopping time. (We will prove this later.)

Counter example: The last time when the process enters a given set is not a stopping time because we need to know the whole future to check it.

Stopping time (cont.)

Lemma 5.3

The

- *sum*
- *maximum*
- *minimum*

of two stopping times is stopping time.

Strong Markov property

Theorem 5.4

Let X_n be Markov chain with transition matrix: $\mathbf{P} = (p(i,j))$ and T be a stopping time. Assuming that $T = n$ and $X_T = y$, every further piece of information about $X_0, \ldots, X_T$ is irrelevant for the future (to estimate values of X_{T+k}) and for $k \geq 0$: X_{T+k} behaves like the original Markov chain started from y.

In the case of $T \equiv k$ we get back the Markov property.

Strong Markov property (cont.)

We only prove now that

$$\mathbb{P}\left(X_{T+1} = z | X_T = y, T = n\right) = p(y, z). \tag{76}$$

For an arbitrary $\mathbf{x} = (x_0, \dots, x_n)$, where $x_i \in S$, let $X_0^n(\mathbf{x})$ be and event defined by

$$X_0^n(\mathbf{x}) = \{X_0 = x_0, \dots, X_n = x_n\}.$$

We define

Strong Markov property (cont.)

$$V_n := \{\mathbf{x} : X_0^n(\mathbf{x}) \Longrightarrow (T = n \text{ and } X_T = y)\}.$$

In other words: V_n is the set of those $\mathbf{x} = (x_0, \dots, x_n)$, for which:

$$X_0 = x_0, \dots, X_n = x_n \Longrightarrow T = n \text{ and } X_T = y$$

Strong Markov property (cont.)

$$
\begin{aligned}
\mathbb{P}\left(X_{T+1}=z, X_T=y, T=n\right) &= \\
&= \sum_{\mathbf{x}\in V_n} \mathbb{P}\left(X_{n+1}=z, X_0^n(\mathbf{x})\right) = \\
&= \sum_{\mathbf{x}\in V_n} \underbrace{\mathbb{P}\left(X_{n+1}=z \mid X_0^n(\mathbf{x})\right)}_{p(y,z)} \cdot \mathbb{P}\left(X_0^n(\mathbf{x})\right) = \\
&= p(y,z) \sum_{\mathbf{x}\in V_n} \mathbb{P}\left(X_0^n(\mathbf{x})\right) = \\
&= p(y,z) \cdot \mathbb{P}\left(T=n, X_T=y\right).
\end{aligned}
$$

We divide both sides by $\mathbb{P}\left(T=n, X_T=y\right)$ and this yields (76).

Recurrent and transient states

Let $T_y^1 := T_y$ and

$$T_y^k := \min\left\{n > T_y^{k-1} : X_n = y\right\}$$

the time of the k^{th} return to y. Because of the strong Markov property

$$\mathbb{P}_y\left(T_y^k < \infty\right) = \rho_{yy}^k.$$

Recurrent and transient states (cont.)

- If $\rho_{yy} < 1$, then the probability of the event that the chain process comes back to y: $\rho_{yy}^k \to 0$. Thus, there's a time when the process no longer gets back to y. These y states are called transient.
- If $\rho_{yy} = 1$. Then for $\forall k$: $\rho_{yy}^k = 1$. Thus the process gets back to y infinitely many times. Then these y states are called recurrent.

Recurrent and transient states (cont.)

The following simple observation will be useful:

Lemma 6.1

If $\mathbb{P}_x(T_y \leq k) \geq \alpha > 0 \; \forall x \in S$, *then*

$$\mathbb{P}_x(T_y > nk) \leq (1-\alpha)^n.$$

Namely, the probability that in the first n steps we have not visited y is less than $1-\alpha$, the same is true for the subsequent $n-1$ blocks of paths of length k.

Recurrent and transient states (cont.)

Definition 6.2

We say that x **communicates with** y ($x \rightsquigarrow y$) if the probability of reaching y from x in some (not necessarily in one) steps is positive. In other words:

$$x \rightsquigarrow y \text{ if } \rho_{xy} = \mathbb{P}_x\left(T_y < \infty\right) > 0.$$

It follows from Markov property that

$$\text{If } x \rightsquigarrow y \text{ and } y \rightsquigarrow z \text{ then } x \rightsquigarrow z. \tag{77}$$

Recurrent and transient states (cont.)

Lemma 6.3

If $\rho_{xy} > 0$ and $\rho_{yx} < 1$, then x is transient.

This is trivial, because since the event {starting from x we can get to y in finitely many steps} has positive probability and the event {from y we don't get back to x} also has positive probability. By Markov property: {starting from x we never get back to x} has also positive probability, so x is transient.

Recurrence and transience

Unless we say otherwise, we do not assume that $\#S < \infty$. Recall:

$$T_y^k = \min\left\{n > T_y^{k-1} : X_n = y\right\}$$

$$\text{and } \rho_{xy} = \mathbb{P}_x\left(T_y < \infty\right).$$

From the strong Markov property:

$$\mathbb{P}_x\left(T_y^k < \infty\right) = \rho_{xy} \cdot \rho_{yy}^{k-1} \tag{78}$$

Recurrence and transience (cont.)

Let

$$N(y) := \# \{n \geq 1 : X_n = y\}.$$

Obviously,

(79) $$\{N(y) \geq k\} = \{T_y^k < \infty\}.$$

Hence, whenever $\rho_{yy} < 1$ (that is y is transient) we have

$$\begin{aligned} \mathbb{E}_x N(y) &= \sum_{k=1}^{\infty} \mathbb{P}_x (N(y) \geq k) = \sum_{k=1}^{\infty} \mathbb{P}_x \{T_y^k < \infty\} \\ &\overset{(78)}{=} \rho_{xy} \sum_{k=1}^{\infty} \rho_{yy}^{k-1} = \frac{\rho_{xy}}{1 - \rho_{yy}} \end{aligned}$$

Recurrence and transience (cont.)

So have obtained that

$$\rho_{yy} < 1 \Longrightarrow \mathbb{E}_x N(y) = \frac{\rho_{xy}}{1 - \rho_{yy}}. \tag{80}$$

That's why $\mathbb{E}_y N(y) < \infty$ iff $\rho_{yy} < 1$. On the other hand we will prove hat

Lemma 6.4

$$\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} p^n(x, y).$$

Recurrence and transience (cont.)

Proof.

$N(y) = \sum_{n=1}^{\infty} \mathbb{1}_{X_n=y}$. Taking expected value:

$$\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} \mathbb{E}_x \left[\mathbb{1}_{X_n=y}\right] = \sum_{n=1}^{\infty} \underbrace{\mathbb{P}_x \left(X_n = y\right)}_{p^n(x,y)} = \sum_{n=1}^{\infty} p^n(x, y).$$

□

Recurrence and transience (cont.)

As a corollary of Lemma 6.4 and (80) we get:

Theorem 6.5

An element $y \in S$ is recurrent if and only if:

$$\sum_{n=1}^{\infty} p^n(y,y) = \mathbb{E}_y\left[N(y)\right] = \infty.$$

Now we prove, using Theorem 6.5 that the **Simple Symmetric Random Walk (SSRW)** on $\mathbb{Z}$ is recurrent. Recall that SSRW is defined on $\mathbb{Z}$ by the transition probability matrix:

$$p(i, i+1) = p(i, i-1) = \frac{1}{2}, \quad \text{for all } i \in \mathbb{Z}.$$

Theorem 6.6

SSRW is null-recurrent on $\mathbb{Z}$. (The same is true on $\mathbb{Z}^2$, but the SSRW is transient in $\mathbb{Z}^d$ for $d \geq 3$.)

We use Stirling-formula in the proof:

(81) $$1 < \frac{n!}{\sqrt{2\pi n} \cdot (n/e)^n} < e^{1/(12n)}.$$

Hence we get

(82) $$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}},$$

where $\sim$ means that the ratio of the two sides tends to 1.

Proof

First we prove that SSRW is recurrent on $\mathbb{Z}$.

Remark: Starting from 0 we get to 0 in $2n$ steps iff we make n steps to the right and n steps to the left. The probability of each of these paths is $(1/2)^{2n}$ and the number of these paths is $\binom{2n}{n}$.

Proof (Cont.)

Hence,

$$p^{2n}(0,0) = \binom{2n}{n}(1/2)^{2n} \sim \frac{1}{\sqrt{\pi n}},$$

where we used the formula given in (82). So,

$$\sum_{n=1}^{\infty} p^n(0,0) \geq \sum_{n=1}^{\infty} p^{2n}(0,0) = \text{const} \cdot \sum_{n=1}^{\infty} n^{-1/2} = \infty.$$

Now we use Theorem 6.5 to conclude that the simple symmetric random walk on $\mathbb{Z}$ is recurrent.

Proof (Cont.)

Now we prove null-recurrence: Let E_k be the expected number of steps required to reach k starting from 0 for the first time. By definition, E_0 is not zero but the expected number of steps of the first return to 0. If we want to get into $k > 1$ from 0, first we have to reach 1, then 2, and so on; and the expected number of getting from i to $i + 1$ is the same for all $i \in \mathbb{Z}$. Hence,

$$E_k = kE_1.$$

Proof (Cont.)

From the 1-step argument:

$$E_1 = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot E_2,$$

because from -1 we can get into 1 in two steps. From this:

$$E_1 = 1 + E_1 \text{ so } \boxed{E_1 = \infty}.$$

Then by the 1-step argumnet we get

$$E_0 = 1 + \frac{1}{2} E_{-1} + \frac{1}{2} E_1,$$

So $\boxed{E_0 = \infty}$, thus the chain is null-recurrent.

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Examples