Markov Chains III

Károly Simon

This course is based on the book:
Essentials of Stochastic processes
by R. Durrett

Department of Stochastics
Budapest University of Technology and Economics
www.math.bme.hu/~simonk

Autumn 2025, BME

Review: exponential distribution

In this chapter we follow [1, Chapter 2].

Review: exponential distribution (cont.)

Definition 1.1

We say that the random variable T has exponential distribution with parameter λ, $T \sim \mathrm{Exp}(\lambda)$ if

$$\mathbb{P}(T \leq t) = 1 - \mathrm{e}^{-\lambda t}, \quad \forall t \geq 0.$$

Equivalently, $T \sim \mathrm{Exp}(\lambda)$ if

$$f_T(t) = \begin{cases} \lambda \mathrm{e}^{-\lambda t}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$$

Review: exponential distribution (cont.)

Some properties of exponential distribution (see [1, Chapter 2]). Let $T \sim \mathrm{Exp}(\lambda)$, $T_i \sim \mathrm{Exp}(\lambda_i)$, $i = 1, \ldots, n$, be independent.

(a) $\mathbb{P}(T > t + s | T > t) = \mathbb{P}(T > s)$.
memoryless property.

(b) $\mathbb{E}[T] = 1/\lambda$ and $\mathrm{Var}(T) = 1/\lambda^2$.

(c) Ha $S \sim \mathrm{Exp}(1)$, then $S/\lambda \sim \mathrm{Exp}(\lambda)$

Review: exponential distribution (cont.)

(d1) $\mathrm{Exp}(\lambda)$ is the only distribution that satisfies the following condition:
(1)
$$\mathbb{P}(t < X < t + \Delta t | X > t) = \lambda \Delta t + \mathfrak{o}(\Delta t),$$

where: $\lim_{\Delta t \to 0} \frac{\mathfrak{o}(\Delta t)}{\Delta t} = 0.$

Review: exponential distribution (cont.)

(d2) Let $T_{\min} := \min\{T_1, \ldots, T_n\}$ and $I \in \{1, \ldots, n\}$ be the index for which $T_I = T_{\min}$. Then $T_{\min}$ and I are independent and $T_{\min} = \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n)$ and

$$\mathbb{P}(I = i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}.$$

Hence,

Review: exponential distribution (cont.)

(e) We are given n alarm clocks of exponential distribution with parameter $\boldsymbol{\mu}$ and let t be a very small number. The probability that the first clock ringings in the time interval $[0, t]$ is approximately $n\mu t$. So,

$$\lim_{t \to 0} \mathbb{P}\left(\min\left\{T_1, \ldots, T_n\right\} < t\right)/t = n\mu.$$

Review: exponential distribution (cont.)

(f) If $T_i = \mathrm{Exp}(\lambda)$, are independent, then distribution of $T = T_1 + \cdots + T_n$ is Gamma distribution with parameter (n, λ). So,

$$f_T(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \text{ if } t \geq 0. \tag{2}$$

Proof is available: [1, page 80]

Review: exponential distribution (cont.)

Recall: Density function of Gamma distribution with parameter X, (α, λ):

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

and $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$.

$$\mathbb{E}[X] = \frac{\alpha}{\lambda} \text{ and } \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}.$$

Review: exponential distribution (cont.)

(g) Let $M_2 := \max\{T_1, T_2\}$. Then

$$M_2 = T_1 + T_2 - \min\{T_1, T_2\}.$$

So, putting together this, (b) and (d2) we get

(3) $$\mathbb{E}[M_2] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.$$

Poisson distribution: review

$X \sim \mathrm{Poi}(\lambda)$, if

$$\mathbb{P}(X = n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}, \text{ if } n = 0, 1, 2, \ldots$$

Properties of Poisson distribution: Let $X \sim \mathrm{Poi}(\lambda)$ and $X_i \sim \mathrm{Poi}(\lambda_i)$, $i = 1, \ldots, n$ be independent. Then

(i) $\mathbb{E}[T] = \mathrm{Var}(T) = \lambda$.

(ii) Let $p(n) \in (0, 1)$: $n \cdot p(n) \to \lambda$ and $Y_n = \mathrm{Binom}(n, p(n))$. Then
$\forall i : \lim_{n \to \infty} \mathbb{P}(Y_n = i) = \frac{\lambda^i}{i!} \cdot e^{-\lambda}$.

(iii) $X_1 + \cdots + X_n = \mathrm{Poi}(\lambda_1 + \cdots + \lambda_n)$.

Poisson process: review

Now we follow [5, Chapter 3]. In this chapter time $t \in [0, \infty)$ is continuous. Let $N(t)$ be the number of customers who enter into a shop until time t. We have three condition about the rate of customers' arrival.

(i) If $I_1, I_2 \subset [0, \infty)$ are disjoint, then the numbers of customers arriving in I_1 and I_2 are independent.

(ii) In an arbitrary small time interval the average number of customers arriving divided by the length of the interval t converges to a constant as $t \to 0$.

(iii) Customers arrive one at a time.

Poisson process: review (cont.)

Inorder to describe this with more mathematical precision first we define the increments of the process $N(t)$.

Let $n \geq 2$ and $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n$. The random variables

$$N(t_1) - N(s_1), \ldots, N(t_n) - N(s_n) \tag{4}$$

are called increments.

(i') $N(0) = 0$ and the increments of $N(t)$ are independent.
(ii')

$$(5) \qquad \mathbb{P}\left(N(t+\Delta t) = N(t)\right) = 1 - \lambda\Delta t + \mathfrak{o}(\Delta t).$$

$$(6) \qquad \mathbb{P}\left(N(t+\Delta t) = N(t)+1\right) = \lambda\Delta t + \mathfrak{o}(\Delta t).$$

(iii')

$$(7) \qquad \mathbb{P}\left(N(t+\Delta t) \geq N(t)+2\right) = \mathfrak{o}(\Delta t).$$

Definition 1.2

If some event (new customer arriving, a phone rings) satisfies conditions (i')-(iii') the number $N(t)$ of events until time t are called $\mathrm{Poisson}(\lambda)$ process. Time intervals between these events (inter event times): $\tau_1, \tau_2, \ldots$. The time when the n^{th} event happens:

$$T_n := \tau_1 + \cdots + \tau_n. \tag{8}$$

So, $N(s) = \max\{n : T_n \leq s\}$.

By definition it is immediate that

Lemma 1.3

1. *$N0) = 0$;*
2. *$t \mapsto N(t)$ is a right continuous function with left limit.*

More precisely: If $y \notin \{T_1, T_2, \dots\}$ then

$$\lim_{\substack{x \to y \\ x < y}} N(x) = \lim_{\substack{x \to y \\ x > y}} N(x) = N(y).$$

On the other hand,

$$\lim_{\substack{x \to T_k \\ x < T_k}} N(x) = k - 1 \text{ and } \lim_{\substack{x \to T_k \\ x > T_k}} N(x) = N(T_k) = k, \quad k \geq 1.$$

Theorem 1.4

(a) *Number of events happened in a fixed-length t_0 time interval $I \subset \mathbb{R}^+$:*

$$\#\{k : T_k \in I\} = \mathrm{Poi}(\lambda \cdot t_0)$$

(b) $\tau_1, \tau_2, \ldots$ *are independent and* $\tau_i = \mathrm{Exp}(\lambda)$.

proof

Part (a) of this Theorem is usually proved in an introductory probability class. Part (b) follows from formulas (5) and (6) by simple computation. Namely, it is obvious that τ_i i.i.d. r.v.

Now we show that

$$\tau_k \sim \mathrm{Exp}(\lambda) \tag{9}$$

Let $x > 0$ and for a large n: $\Delta t = \frac{x}{n}$. Let $y \geq 0$ be arbitrary and

$$F_\ell(x) := \mathbb{P}\left(T_\ell \leq y + x \middle| T_{\ell-1} = y\right).$$

proof (cont.)

Now we partition $[y, y+x]$ as follows:

$$\{I_k = [y + (k-1)\Delta t, y + k\Delta t]\}_{k=1}^{n}.$$

Let $F_\ell(x)$ be the sum of those probabilities when the ℓ^{th} event happens in I_k. So:

$$\begin{aligned} F_\ell(x) &= \sum_{k=0}^{n-1} \lambda\Delta t(1-\lambda\Delta t)^k + n\cdot\mathfrak{o}(\Delta t) \\ &= \lambda\Delta t \sum_{k=0}^{n-1}(1-\lambda\Delta t)^k + n\cdot\mathfrak{o}(\Delta t) \\ &= 1-(1-\lambda\Delta t)^n + n\cdot\mathfrak{o}(\Delta t) \\ &= 1-\left(1-\lambda\frac{x}{n}\right)^n + n\cdot\mathfrak{o}\left(\frac{x}{n}\right) \end{aligned}$$

proof (cont.)

Using that x is fixed, the last part converges to 0 if $n \to \infty$. Thus

$$F_\ell(x) = 1 - e^{-\lambda x}. \tag{10}$$

If $\ell = 1$, applying the above reduction, for $T_0 = 0$ we get that $\tau_1 = \mathrm{Exp}(\lambda)$. Now from induction and with the law of Total Probability, we get that $\tau_\ell \sim \mathrm{Exp}(\lambda)$. The independence of $\{\tau_1, \tau_2, \dots\}$ also comes from formula (10).

In summary

Summarizing what we have seen:

Theorem 1.5

The non-negative integer valued stochastic process $\{N(s) : s \geq 0\}$ *is a* $\mathrm{Poisson}(\lambda)$ *process if and only if*

a. $N(0) = 0$,
b. $N(t + s) - N(s) = \mathrm{Poi}\,(\lambda t)$,
c. $N(t)$ *has independent increments.*

Poisson processes

Theorem 1.6

Let $t_0 > 0$ *be arbitrary. Assume that in the time interval* $[0, t_0]$ *exactly one event of a Poisson process happened. Then the distribution of the time when this event happened is uniform in the interval* $[0, t_0]$.

Proof

Let $0 \leq s \leq t_0$ and $\mathfrak{P} := \mathbb{P}\left(\tau_1 \leq s | X_{t_0} = 1\right)$. Then

Poisson processes (cont.)

Proof (cont.)

$$\begin{aligned}
\mathfrak{P} &= \frac{\mathbb{P}\left(\{\tau_1 \leq s\} \cap \{N(t_0) = 1\}\right)}{\mathbb{P}\left(N(t_0) = 1\right)} \\
&= \frac{\mathbb{P}\left(\{N(s) = 1\} \cap \{N(t_0) = 1\}\right)}{\mathbb{P}\left(N(t_0) = 1\right)} \\
&= \frac{\mathbb{P}\left(\{N(s) = 1\} \cap \{N(t_0) - N(s) = 0\}\right)}{\mathbb{P}\left(N(t_0) = 1\right)} \\
&= \frac{\mathbb{P}\left(\{N(s) = 1\}\right) \cdot \mathbb{P}\left(\{N(t_0 - s) = 0\}\right)}{\mathbb{P}\left(N(t_0) = 1\right)} \\
&= \frac{(\lambda s)e^{-\lambda s} \cdot e^{-\lambda(t_0 - s)}}{(\lambda t_0)e^{-\lambda t_0}} = \frac{s}{t_0}.
\end{aligned}$$

The following theorem can be proven in a similar way: (see [7, page 126]):

Theorem 1.7

Let $0 = s_0 < s_1 \leq \cdots < s_n < t$ *and let*

$$F(s_1, \dots, s_n) := \mathbb{P}\left(T_1 \leq s_1, \dots, T_n \leq s_n | N(t) = n\right).$$

Then

$$F(s_1, \dots, s_n) = \frac{n!}{t^n} \prod_{j=1}^{n} \left(s_j - s_{j-1}\right). \tag{11}$$

This complicated expression can be reformulated in the following way:

Let

$$U_1, \dots, U_n$$

be independent $\mathrm{Uniform}[0, t]$ r.v.. We arrange them in increasing order:

$$V_1 < \cdots < V_n.$$

Theorem 1.8

Assuming that $N(t) = n$,

$$(T_1, \dots, T_n) \stackrel{d}{=} (V_1, \dots, V_n),$$

where $V_1, \dots, V_n$ were defined above.

Conditioning

In summary: what we have seen above it says that:

Assuming that we have n arrivals by time t,

the locations of these n arrivals are the same as

the location of n points thrown uniformly on the time interval $[0, t]$.

This implies that

Conditioning (cont.)

Theorem 1.9

Assume that $s < t$ *and* $0 \leq m \leq n$ *Then*

$$\mathbb{P}\left(N(s) = m | N(t) = n\right) = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}.$$

Poisson processes: two weaknesses

Problem 1.10

How many students arrive at Campus Canteen between 11:00 and 13:00?

Can we use a Poisson process with a suitable parameter to model this situation? To answer this, first we must check conditions (i)-(iii) of the definition of Poisson processes on slide 13.

Here we face with two serious problems:

Poisson processes: two weaknesses (cont.)

(a) By (ii) from slide 13: in a fixed tiny time interval the number of arriving students is approximately the length of the interval multiplied by a constant.

(b) By (iii) 13: students come one by one at a time.

Poisson processes: two weaknesses (cont.)

As opposed to that at times when great lectures end, much more students go to the canteen than in any other times. So, ((a) is not satisfied). Moreover, often students go to canteen together with friends ((b) is not satisfied). Fortunately, we can use a variant of Poisson process for answering the question above with the following modifications:

- To tackle the problem mentioned in (a) we use non-homogeneous Poisson process.

Poisson processes: two weaknesses (cont.)

- To deal with the problem mentioned in (b) we introduce Compund Poisson processes.

Definition 1.11 (non-homogeneous Poisson process)

$\{N(s) : s \geq 0\}$ is non-homogeneous Poisson process with rate $\lambda(r)$ if

1. $N(0) = 0$,
2. increments of $N(t)$ are independent,
3. $N(t) - N(s) = \mathrm{Poi}\left(\int_s^t \lambda(r)dr\right)$.

The meaning of (3): $\forall t \in [0, \infty)$, $\forall \Delta > 0$:

$$\mathbb{P}(N(t+\Delta) - N(t) = 0) = 1 - \lambda(t)\Delta + \mathfrak{o}(\Delta)$$
$$\mathbb{P}(N(t+\Delta) - N(t) = 1) = \lambda(t)\Delta + \mathfrak{o}(\Delta)$$
$$\mathbb{P}(N(t+\Delta) - N(t) \geq 2) = \mathfrak{o}(\Delta)$$

Warning: In this case, unlike in the case of Poisson processes , the inter even times $\tau_1, \tau_2, \ldots$:

- $\tau_1, \tau_2, \ldots$ are NOT independent,
- τ_i, $i = 1, 2, \ldots$ do NOT have exponential distribution.

Namely: let $\mu(t) := \int_0^t \lambda(x)dx$. Let f_{τ_1} be the density function of τ_1 and let f_{τ_1,τ_2} be the joint density function of (τ_1, τ_2), then simple calculation shows that

1. $f_{\tau_1}(t) = -\frac{d}{dt}\mathbb{P}\left(\tau_1 > t\right) = \lambda(t)\mathrm{e}^{-\mu(t)}$.
2. $f_{\tau_1,\tau_2}(s,t) = \lambda(s)\mathrm{e}^{-\mu(s)} \cdot \lambda(s+t)\mathrm{e}^{-(\mu(s+t)-\mu(s))}$.

Hence, if $\lambda(r)$ is not constant, then τ_1 is not exponential and τ_1, τ_2 are not independent.

Compound Poisson processes

Example 1.12 (Motivating examle)

It is reasonable to assume that at McDonald's drive-through section between $12:00$ and $13:00$ the number of arriving cars is $\sim \mathrm{Poisson}(\lambda)$. Let $N(t)$ be the cars arriving till time $12+t$. Let Y_i be the number of customers in the i^{th} car. We can assume, that Y_i i.i.d. (independent identically distributed) and that Y_i is independent of the arrival times. Then number of customers until time t

$$S(t) = Y_1 + \cdots + Y_{N(t)}.$$

Theorem 1.13

Let $Y_1, Y_2, \ldots$ be i.i.d. r.v. and we are also given a non-negative integer valued r.v. N which is independent of $\{Y_i\}_i$. Let

$$S := Y_1 + \cdots + Y_N.$$

Then

1. *If* $\mathbb{E}[Y_i], \mathbb{E}[N] < \infty \Rightarrow \mathbb{E}[S(t)] = \mathbb{E}[N(t)] \cdot \mathbb{E}[Y_i]$.
2. *If* $\mathbb{E}\left[Y_i^2\right], \mathbb{E}\left[N^2\right] < \infty \Rightarrow$
   $\mathrm{Var}(S(t)) = \mathbb{E}[N(t)] \mathrm{Var}(Y_i) + \mathrm{Var}(N(t)) (\mathbb{E}[Y_i])^2$.
3. *If* $N = \mathrm{Poisson}(\lambda) \Rightarrow \mathrm{Var}(S(t)) = \lambda t \mathbb{E}\left[Y_i^2\right]$.

Thinning

Let $N(t) = \mathrm{Poisson}(\lambda)$ and let us associate i.i.d. non-negative integer valued r.v. Y_i to the i^{th} event, such that Y_i are independent of $N(t)$. We define

$$N_j(t) := \#\left\{i \le N(t) : Y_i = j\right\}. \tag{12}$$

In Example 1.12, where Y_i is the number of people in the i-th car, $N_j(t)$ is the number of cars with j passengers that arrive before time t.

Thinning (cont.)

Theorem 1.14

Let $N_j(t)$ as on slide 37. Then $N_j(t)$ are independent Poisson processes with rate:
$N_j(t) = \text{Poisson}(\lambda \cdot \mathbb{P}(Y_i = i))$.

The proof can be found at [1, Section 2.4]. This method is called thinning a Poisson process because here we take a Poisson process and split into more than one Poisson processes.

Thinning (cont.)

Example 1.15

Customers arrive in a bank according $N(t) = \text{Poisson}(\lambda)$. A wicked boy sitting close to the entrance, tosses a bias coin which lands on head with probability $1/3$ whenever a new customer arrives. Whenever the coin lands head the wicked boy pours a glass of water at the newly arrived customer. If the coins lands on tail the boy does not pour water at the newly arrived customer. Let $W(t)$ and $D(t)$ be the number of customers who arrived at the bank by time t and get wet and remained dry respectively. Then $D(t)$ and $W(t)$ are independent and $W(t) = \text{Poisson}\,(\lambda/3)$ and $D(t) = \text{Poisson}\,(2\lambda/3)$.

Example

Example 1.16 (Example for Thinning of a Poisson process)

Assume that the arrival of customers into a bank is given by a Poisson process of rate 10 per hour $N(t) \sim \text{Poisson}(10)$. Moreover, the distribution of the gender (male or female) of a customer is $(1/2, 1/2)$ independently of everything. $\mathbb{P}(Y_i = \text{Male}) = \mathbb{P}(Y_i = \text{Female}) = 1/2$. Let $N_F(t)$ and $N_M(t)$ be the number of female and male costumers arrived by time t. Then $N_F(t)$ and $N_M(t)$ are independent Poisson processes with rate 5.

Thinning of the nonhomogeneous Poisson Processes

One can easily extend the previous result for the case of the nonhomogeneous Poisson processes:

Theorem 1.17

Given a Poisson process with rate λ. We retain a point that lands at time s with probability $p(s)$ and we throw it away with probability $1 - p(s)$. Then $N(t)$ number of points that we retained by a time t results is a nonhomogeneous Poisson process with rate $\lambda p(s)$.

Thinning of the nonhomogeneous Poisson Processes (cont.)

Example 1.18 ($M/G/\infty$ queue)

- (a) There are infinitely many telephone lines,
- (b) Beginnings of calls follow Poisson process,
- (c) Let G be the cumulative distribution function of the length T of the calls: $G(t) := \mathbb{P}(T \leq t)$. We assume that $G(0) = 0$ and $\mathbb{E}[T] = \mu$.

Input: is a Poisson process that is Markov and the service time is General and there are infinitely many queues. Question: the number of calls in the system after long time.

Thinning of the nonhomogeneous Poisson Processes (cont.)

First we assume that the systems is empty at time zero. Consider a call that started at time s. Then it has been finished by time t with probability $G(t-s)$. Hence, the probability that a call started at time s is still in progress is $1-G(t-s)$. (This was probability $p(s)$ in Theorem 1.17). So, by Definition1.11, the number of calls in progress at time t: is a Poisson distribution with rate:

$$\int_{s=0}^{t} \lambda(1-G(t-s))ds = \lambda \int_{r=0}^{t}(1-G(r))dr \tag{13}$$

Thinning of the nonhomogeneous Poisson Processes (cont.)

Letting $t \to \infty$:

$$\lambda \int_{r=0}^{\infty} (1 - G(r)) dr = \lambda \mu. \tag{14}$$

Namely, if T is the length of the calls then

$$\mu = \mathbb{E}[T] = \int_{r=0}^{\infty} \mathbb{P}(T \geq r)\, dr = \int_{r=0}^{\infty} (1 - G(r)) dr.$$

This means that the average number of the calls in the system is the product of the rate at which calls enter times the average duration of the calls.

Superposition of Poisson processes

Theorem 1.19

Suppose $N_1(t), \dots N_\ell(t)$ are independent Poisson processes with rates

$$\lambda_1, \dots, \lambda_\ell,$$

then

$$N_1(t) + \cdots + N_\ell(t)$$

is a Poisson process with rate $\lambda_1 + \cdots + \lambda_\ell$.

For the proof see [1, Setion 2.4.2]

Poisson Race

Example 1.20

Given a Poisson process of red arrivals with rate λ and an independent Poisson process of green arrivals with rate μ, what is the probability that we will get 6 red arrivals before a total of 4 green ones?

An observation:

It is easy to see that following event happens

$$E := \{\text{at least 6 red arrivals in the first 9 arrivals}\}.$$

iff we have 6 red arrivals before a total of 4 green ones.

Poisson Race (cont.)

The idea of the Solution:

In virtue of Theorem 1.20 we can look at this process as running a usual Poisson process of parameter $\mu + \lambda$ (the total arrivals of green and red together) and thinning it to get the process of green arrivals and the process of red arrivals. Namely, each arrival of the $\mathrm{Poisson}(\lambda + \mu)$ process is painted red with probability $p := \frac{\lambda}{\lambda+\mu}$ and painted green with probability $1 - p = \frac{\mu}{\lambda+\mu}$. These painted process are independent $\mathrm{Poisson}(\lambda)$ and $\mathrm{Poisson}(\mu)$ processes respectively.

Poisson Race (cont.)

Solution

Recall that $p := \frac{\lambda}{\lambda+\mu}$. Then

$$\mathbb{P}(E) = \sum_{k=6}^{9} \binom{9}{k} p^k (1-p)^{9-k}.$$

If $\lambda = \mu = \frac{1}{2}$ then $\mathbb{P}(E) = \frac{140}{512} = 0.273$.

Examples

[1] R. Durrett
Essentials of Stochastic Processes, Second edition
Springer, 2012. Click here to download from the authors website the almost-ready version.

[2] R. Durrett
Probability: Theory and Examples
2019. Click here to download from the authors website

[3] S. Karlin, H.M. Taylor
A first course in stochastic processes
Academic Press,2014

[4] S. Karlin, H.M. Taylor
A second course in stochastic processes
, Academic Press, 1981

[5] G. Lawler
Intoduction to Stochastic Processes
Chapmann & Hall 1995.

[6] D.A. Levin, Y. Peres,E.L. Wilmer
Markov chains and mixing times
American Mathematical Society, 2009.

[7] Piet Van Mieghem
The Poisson process
http://www.nas.its.tudelft.nl/people/Piet/CUPbookChapters/PACUP_Poisson.pdf