

Brownian motion

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This course is based on the book:
Essentials of Stochastic processes
by R. Durrett

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1 Normal distribution, Gaussian process

2 Brownian motion

Definition 1.1 (Normal distribution (on $\mathbb{R}$))

Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Random variable X with parameters (μ, σ^2) **has normal** (or Gaussian) distribution $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density function:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

If $\mu = 0$ and $\sigma = 1$, then we get the **standard normal distribution** $\mathcal{N}(0, 1)$. We use the following notation this section:

$$(1) \quad \varphi(x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}, \quad \Phi(x) := \int_{-\infty}^x \varphi(y) dy.$$

Some properties

$X \sim \mathcal{N}(\mu, \sigma^2)$ and $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, 2$. Then

(a) $\mathbb{E}[X] = \mu$, $\text{Var}(X) = \sigma^2$.

(b) $F_X(x) = \mathbb{P}(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

(c) $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(d) $X \sim \mathcal{N}(0, 1)$, then

$$(2) \quad \frac{1}{\sqrt{2\pi}} \cdot (x^{-1} - x^{-3}) \cdot e^{-x^2/2} \leq \mathbb{P}(X \geq x) \leq \frac{1}{\sqrt{2\pi}} \cdot x^{-1} \cdot e^{-x^2/2}$$

(d) $Y \sim \text{Bin}(n, p)$, $a < b$, then

$$(3) \quad \lim_{n \rightarrow \infty} \mathbb{P}\left(a < \frac{Y - np}{\sqrt{np(1-p)}} < b\right) = \Phi(b) - \Phi(a).$$

Multivariate normal distribution

This is a review of material taught in the course Probability 1. Let A be a matrix of $d \times d$, which is **symmetric**, **positive definit** and $\mathbf{m} \in \mathbb{R}^d$ be a fixed vector. A random variable $\mathbf{X}$ which takes values in $\mathbb{R}^d$ has multivariate normal or Gaussian distribution, if its density function is of the form:

$$f(\mathbf{x}) = \frac{\sqrt{\det(A)}}{(2\pi)^{d/2}} \cdot e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \cdot A \cdot (\mathbf{x}-\mathbf{m})}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Multivariate normal distribution (cont.)

The meaning of the matrix A : Let $\mathbf{X} = (X_1, \dots, X_d)$.

$$(A^{-1})_{ij} = \mathbf{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[X_i]) \cdot (X_j - \mathbb{E}[X_j])].$$

Multivariate normal distribution (cont.)

Definition 1.2

Let $\mathbf{X}$ be as above. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of A , and $\mathbf{v}_1, \dots, \mathbf{v}_d$ be the **ortonormal basis** of $\mathbb{R}^d$ with the appropriate eigenvectors. Let us define diagonal matrix

$$D := \text{diag}(\lambda_1, \dots, \lambda_d).$$

We form the $d \times d$ orthogonal matrix:

$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_d \end{bmatrix}$ by eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ as column vectors.

Multivariate normal distribution (cont.)

Lemma 1.3

Let $\mathbf{X}$ be as above. Then

$$(4) \quad \mathbf{X} = P \cdot D^{-1/2} \cdot (Y_1, \dots, Y_d) + \mathbf{m}$$

where $Y_i = \mathcal{N}(0, 1)$, $i = 1, \dots, d$ and they are all independent.

See [1, chapters 6 and 7].

1 Normal distribution, Gaussian process

2 Brownian motion

Definition of the one-dimensional Brownian motion

(a) If $0 \leq t_0 < t_1 < \cdots < t_n$ are positive numbers, then

$$B(t_0), B(t_1) - B(t_0), \dots, B(t_n) - B(t_{n-1})$$

are independent. (Independent increments.)

(b) If $s, t \geq 0$, then

$$\mathbb{P}(B(s+t) - B(s) \in A) = \int_A (2\pi t)^{-1/2} \exp\left(-\frac{x^2}{2t}\right) dx$$

Definition of the one-dimensional Brownian motion (cont.)

(Stationary increments.)

(c) With probability one: $t \rightarrow B_t$ is continuous.

Constructing Brownian motion

Let $\mathcal{D}_0 := \{0, 1\}$ and for some $n \geq 1$:

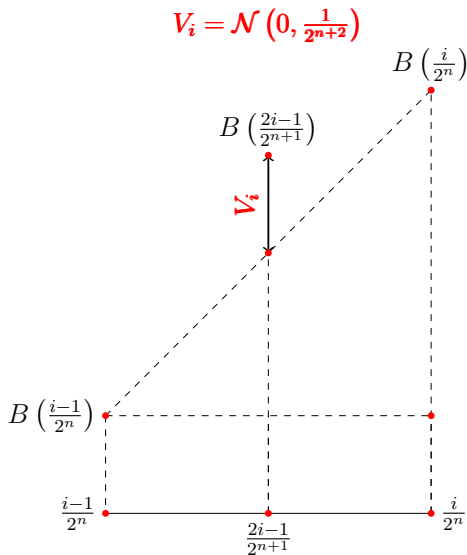
$$\mathcal{D}_n := \left\{ \frac{2k-1}{2^n} : 1 \leq k \leq 2^{n-1}, k \text{ is odd} \right\}$$

$$\mathcal{D} := \bigcup_{n=0}^{\infty} \mathcal{D}_n$$

is the set of dyadic rational numbers in $[0, 1]$.

Construction of random function $B : [0, 1] \rightarrow \mathbb{R}$:

Constructing Brownian motion (cont.)



Constructing Brownian motion

- $B(0) := 0$, $B(1) = \mathcal{N}(0, 1)$.
- Let us assume that $B\left(\frac{i}{2^n}\right)$, $0 \leq i \leq 2^n$ have already been defined. Fix a $t \in \mathcal{D}_{n+1}$. Then for some $1 \leq i \leq 2^n$:

$$t = \frac{2i-1}{2^{n+1}} = \frac{1}{2} \left(\frac{i-1}{2^n} + \frac{i}{2^n} \right),$$

where $B\left(\frac{i-1}{2^n}\right)$ and $B\left(\frac{i}{2^n}\right)$ have already been defined.
Let:

(5)

$$B(t) := \mathcal{N} \left(\frac{1}{2} \cdot \left(B\left(\frac{i-1}{2^n}\right) + B\left(\frac{i}{2^n}\right) \right), \frac{1}{2^{n+2}} \right).$$

Constructing Brownian motion (cont.)

- for $x \in (0, 1)$ let

$$(6) \quad B(x) := \lim_{t \uparrow x, t \in \mathcal{D}} B(t) = \lim_{t \downarrow x, t \in \mathcal{D}} B(t).$$

To see that this definition is correct, we must prove that the limits above exist and if $x \in \mathcal{D}$, then these limits are equal to $B(x)$.

Theorem 2.1

$n \geq 1$ is fixed. For $i = 1, \dots, 2^n$:

$$(7) \quad Z_i := B\left(\frac{i}{2^n}\right) - B\left(\frac{i-1}{2^n}\right).$$

Then

(a) $\{Z_i\}_{i=1}^{2^n}$ are independent,

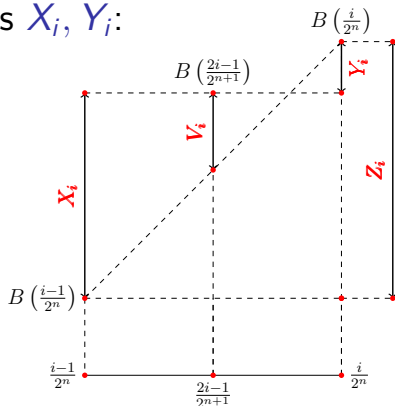
(b) $\forall i: Z_i \in \mathcal{N}\left(0, \frac{1}{2^n}\right)$.

(c) If $s, t \in \mathcal{D}$, $s < t$, then

$$(8) \quad B(t) - B(s) \in \mathcal{N}(0, s - t).$$

Proof I

Let us assume that for some n the statement is true. We have defined variables V_i and Z_i for $i = 1, \dots, 2^n$. We use induction to prove the theorem. Let us introduce random variables X_i, Y_i :



Proof II

We know that the following random variables are independent:

$$Z_1, \dots, Z_{2^n}, V_1, \dots, V_{2^n}.$$

Their joint density function at $(\mathbf{z}, \mathbf{v}) = (z_1, \dots, z_{2^n}, v_1, \dots, v_{2^n})$ is as follows

$$f_{\mathbf{z}, \mathbf{v}}(\mathbf{z}, \mathbf{v}) = \prod_{i=1}^{2^n} \frac{2^{n/2}}{\sqrt{2\pi}} \exp\left(-\frac{z_i^2 2^n}{2}\right) \cdot \frac{2^{(n+2)/2}}{\sqrt{2\pi}} \exp\left(-\frac{v_i^2 2^{n+2}}{2}\right)$$

We have just used the inductive assumption and that $\text{Var}(Z_i) = \frac{1}{2^n}$. $\text{Var}(V_i) = 2^{-(n+2)}$ by definition (see (5)).

Proof III

Let $G : \mathbb{R}^{2^n} \times \mathbb{R}^{2^n} \rightarrow \mathbb{R}^{2^n} \times \mathbb{R}^{2^n}$

$$(9) \quad (\mathbf{x}, \mathbf{y}) = G(\mathbf{z}, \mathbf{v}) := \left(\frac{1}{2}\mathbf{z} + \mathbf{v}, \frac{1}{2}\mathbf{z} - \mathbf{v} \right).$$

The same coordinate wise: for $1 \leq i \leq 2^n$:

$$(10) \quad x_i := \frac{1}{2}z_i + v_i, \quad y_i := \frac{1}{2}z_i - v_i.$$

Using formula (20) from Appendix:

$$f_{\mathbf{X}, \mathbf{Y}}(\mathbf{x}, \mathbf{y}) = f_{\mathbf{Z}, \mathbf{V}}(G^{-1}(\mathbf{x}, \mathbf{y})) \cdot |\det G'(G^{-1}(\mathbf{x}, \mathbf{y}))|^{-1},$$

where $(\mathbf{z}, \mathbf{v}) = G^{-1}(\mathbf{x}, \mathbf{y})$; with coordinates:
 $z_i = x_i + y_i$ and $v_i = \frac{1}{2}(y_i - x_i)$.

Proof IV

Writing these into the definition of $f_{\mathbf{z}, \mathbf{v}}(\mathbf{z}, \mathbf{v})$ we get that the joint density function of random variables

$$(X_1, \dots, X_{2^n}, Y_1, \dots, Y_{2^n})$$

is

$$\prod_{i=1}^{2^n} \frac{2^{(n+1)/2}}{\sqrt{2\pi}} \exp\left(-\frac{x_i^2 2^{n+1}}{2}\right) \cdot \frac{2^{(n+1)/2}}{\sqrt{2\pi}} \exp\left(-\frac{y_i^2 2^{n+1}}{2}\right)$$

This means that

$$X_1, \dots, X_{2^n}, Y_1, \dots, Y_{2^n}$$

are independent random variables of $\mathcal{N}(0, 1/2^{n+1})$. From this, proving part (c) is trivial. ■

Equivalent definition

- (i) $B(t)$ is a Gaussian process (all of its finite dimensional distributions are Gaussian (normal)).
- (ii) $\mathbb{E}[B_s] = 0$ and $\mathbb{E}[B_s \cdot B_t] = s \wedge t := \min\{s, t\}$.
- (iii) With probability one: $t \rightarrow B_t$ is continuous.

Basic properties

(a) Let $T, \lambda > 0$. Then

$$\mathbb{P} \left(\sup_{t \in [0, T]} |B_t| \geq \lambda \right) \leq \frac{T}{\lambda^2}.$$

(b) Scaling independence:

$$t \rightarrow a^{-1/2} B(at)$$

is also a Brownian motion.

(c) time reversal:

$$t \rightarrow t B(1/t)$$

is also a Brownian motion.

Basic properties II

(d) Law of the iterated logarithm:

$$(11) \quad \limsup_{t \rightarrow \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = 1 \quad \text{a.s.}$$

- (e) $B(t)$ is Hölder for every $\alpha < \frac{1}{2}$ for class α , but not for $\alpha = \frac{1}{2}$.
- (f) The trajectories of Brownian motion are nowhere differentiable almost surely.

Mirroring Theorem

Let τ be the time when the Brownian motion starting from zero first reaches the previously fixed number a . Let

$$\widehat{B} := \begin{cases} B(t), & \text{ha } t < \tau; \\ a - (B(t) - a), & \text{ha } t > \tau. \end{cases}$$

We get the graph of it by mirroring graph of $B(t)$ to the horizontal line in interval $t > \tau$. $\widehat{B}(t)$ is also a Brownian motion.

Brownian motion in $\mathbb{R}^d$

Definition 2.2

- Brownian motion in $\mathbb{R}^d$ is a random continuous function $\mathbf{B} : [0, \infty) \rightarrow \mathbb{R}^d$ with the following properties:
- (a) $\mathbf{B}(0) = \mathbf{0}$.
 - (b) for $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$: $\{\mathbf{B}(t_i) - \mathbf{B}(t_{i-1})\}_{i=1}^n$ are independent.
 - (c) for $0 \leq s < t$: density function of $\mathbf{B}(t) - \mathbf{B}(s)$:

$$(12) \quad (2\pi(t-s))^{-d/2} \cdot \exp\left(-\frac{\|\mathbf{x}\|^2}{2(t-s)}\right).$$

Brownian motion in $\mathbb{R}^d$ (cont.)

Lemma 2.3

Let $B_1(t), \dots, B_d(t)$ be independent one-dimensional Brownian motions. Then

$$(13) \quad \mathbf{B}(t) := (B_1(t), \dots, B_d(t))$$

is a d -dimensional Brownian motion.

Proof: Density function of $\mathbf{B}(t) - \mathbf{B}(s)$ has form (12).

Namely: density function of $B_i(t)$:

$f_i(x_i) = (2\pi(t-s))^{-1/2} \exp\left(-\frac{x_i^2}{2(t-s)}\right)$. For components

Brownian motion in $\mathbb{R}^d$ (cont.)

of B_i are independent, density function of $\mathbf{B}(t) - \mathbf{B}(s)$ is $\prod_{i=1}^n f_i(x_i)$, which has form (12). The remaining part of the proof is even simpler.

Hölder continuity

Theorem 2.4

The d -dimensional Brownian motion is Hölder-continuous for every $\alpha < \frac{1}{2}$. In other words, there exist constants $\varepsilon > 0$ and $c = c(d, \alpha)$ so that for all $|h| < \varepsilon$, $t \geq 0$ and $t + h \geq 0$:

$$|\mathbf{X}(t + h) - \mathbf{X}(t)| \leq c \cdot h^\alpha.$$

Appendix: Integration by substitution

Let X, Y be metric spaces and $G : X \rightarrow Y$ be a mapping. Let us assume that μ is a measure on X . Then let us define the "push forward" measure $G_*\mu$ on set Y :

$$G_*\mu(B) := \mu(G^{-1}(B)).$$

Appendix: Integration by substitution (cont.)

Theorem 2.5

Let us assume that

- U, V are separable metric spaces,
- $G : U \rightarrow V$ is Borel-measurable,
- μ is a Borel-measure on U ,
- $f : V \rightarrow [0, \infty)$,

Then

$$(14) \quad \int_V f(v) d(G_*\mu)(v) = \int_U (f \circ G)(u) d\mu(u).$$

We use this theorem in the special case when

Appendix: Distribution transformations

Let X, Y be continuous random variables in $\mathbb{R}^d$, for which

- density function of X, Y : f_X, f_Y
- $Y = G(X)$, where $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ C^1 is bijection.

Appendix: Distribution transformations (cont.)

Then for every Borel set $H \subset \mathbb{R}$:

$$\begin{aligned}\int_{x \in H} f_X(x) dx &= \mathbb{P}(X \in H) = \mathbb{P}(Y \in G(H)) \\ &= \int_{y \in G(H)} f_Y(y) dy \\ &= \int_{x \in H} f_Y(G(x)) \cdot |\det(G'(x))| dx,\end{aligned}$$

Appendix: Distribution transformations (cont.)

where in the last step we have used formula (18) of Integration by substitution. Using that this holds for all Borel sets, we get:

$$(19) \quad f_X(x) = f_Y(G(x)) \cdot |\det(G'(x))|.$$

Applying substitution $y = G(x)$ on this, we get that

$$(20) \quad f_Y(y) = f_X(G^{-1}(y)) \cdot |\det G'(G^{-1}(y))|^{-1}.$$

Definition of Stochastic process

A probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and an index set T are fixed. Let $X_t, t \in T$ be random variables in probability space $(\Omega, \mathcal{A}, \mathbb{P})$. $X_t, t \in T$ is called **stochastic process**. We can think of $\{X_t\}$ as the path of a particle moving randomly in the state space S . The position of the particle at time t is given by $X_t \in S$. In the terms of this course, a stochastic process is defined, if given its:

- state space
- index set
- finite dimensional distributions.

Definition of Stochastic process (cont.)

These don't completely determine the stochastic process if the parameter is continuous.

Example: Let $B(t)$ the standard Brownian motion on the line and U be an independent uniform r.v. on $[0, 1]$. Furthermore:

$$\widehat{B}(t) := \begin{cases} B(t), & t \neq U; \\ 0, & t = U. \end{cases}$$

Then the finite dimensional distributions of $\widehat{B}(t)$ and $B(t)$ are the same but trajectories of $\widehat{B}(t)$ are not continuous.

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