Random Frcatals

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This course is based on the book:
Essentials of Stochastic processes
by R. Durrett

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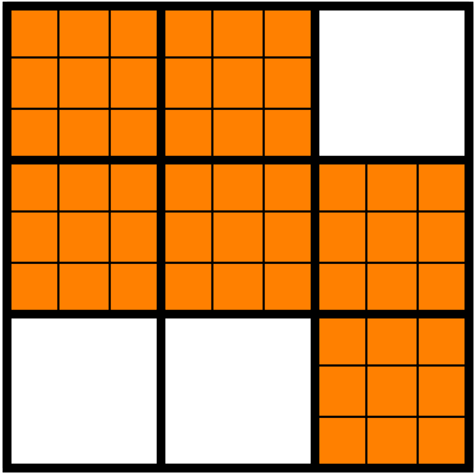
We are given a bias coin:

$P\Big($  $\Big) = p,$

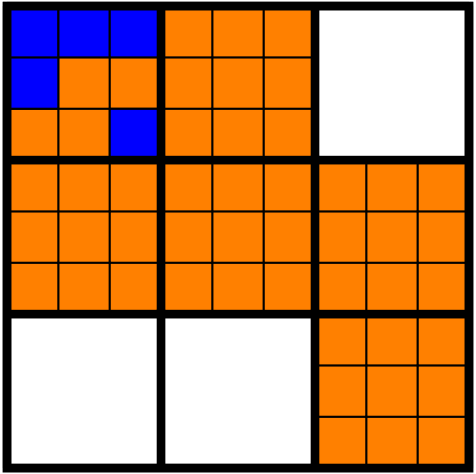
$P\Big($  $\Big) = 1 - p$

We always flip the coin independently of everything.

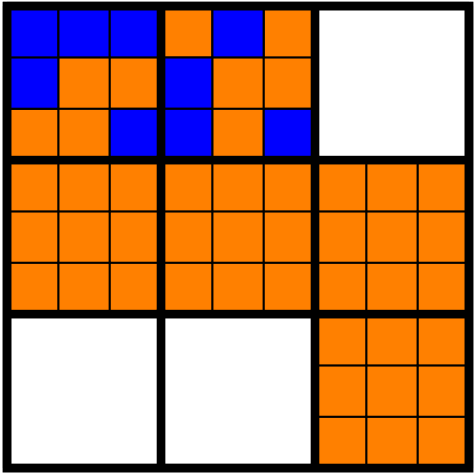


Λ_1


JAMES MADISON
4th PRESIDENT 1809-1817
UNITED STATES OF AMERICA
$1



Λ_2



Λ_2


JAMES MADISON
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$1
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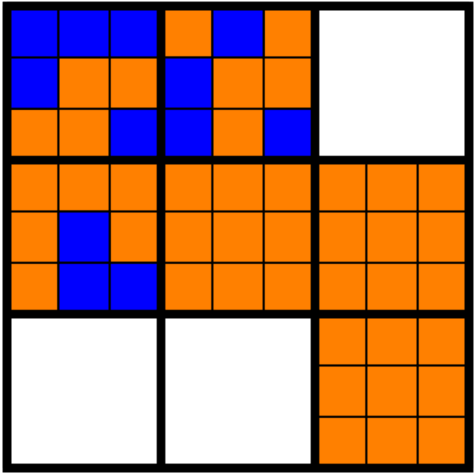
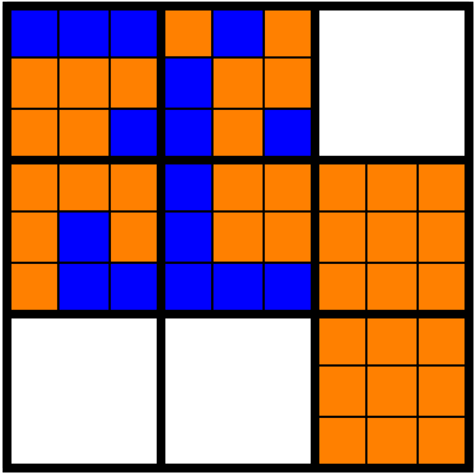


Λ_2



Λ_2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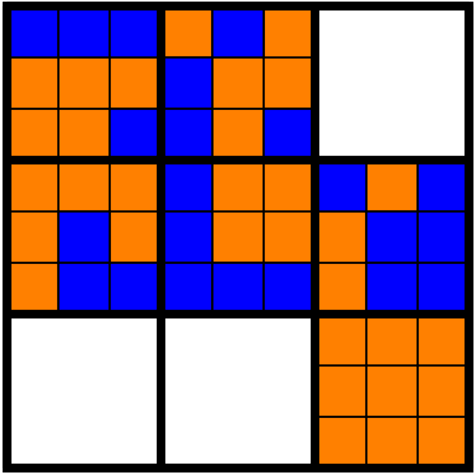
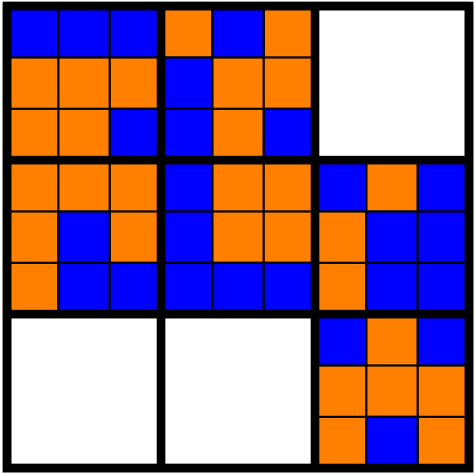


Λ_2



Λ_2

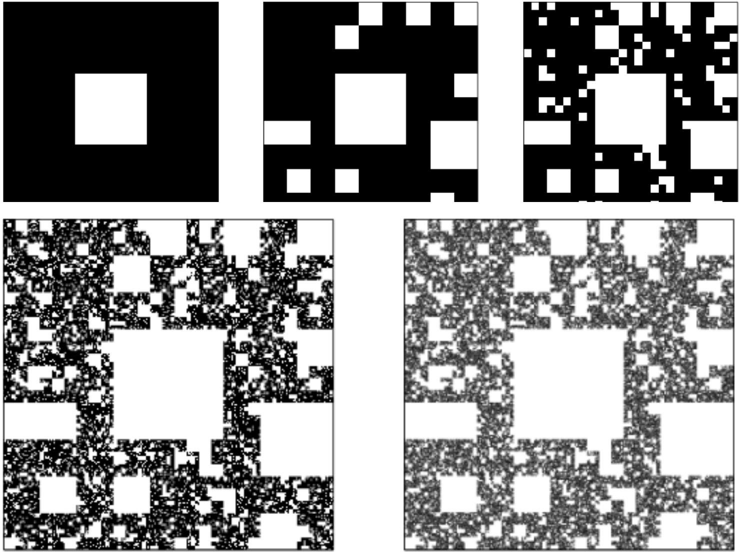


Figure: The first 5 approximation $M = 3$, $p = 0.85$

Fractal percolation (or Mandelbrot percolation) on the unit square

Let $\mathcal{E}_n$ be the set of retained level n squares. We write

$$\Lambda_n := \bigcup_{Q \in \mathcal{E}_n} Q.$$

Then the statistically self-similar set of interest is, which is called fractal percolation set

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

The `Fractal percolation` (or Mandelbrot percolation) random set $\Lambda_{M,p}^d$ on $[0,1]^d$ with parameters (M,p) is obtained if we divide the unit cube $[0,1]^d$ into M^d congruent cubes of size $1/M$ and apply the process described above.

Above we described the construction in the special case when $d = 2$ and $M = 3$.

The event $\Lambda_{M,p}^d = \emptyset$ is called Extinction. This happens with positive probability if $p < 1$ (what we always assume). So, most of our assertions will be conditioned on no-extinction which means that $\Lambda_{M,p}^d \neq \emptyset$

It was proved by Falconer and independently Mauldin, Williams that conditioned on non-extinction:

(1) $$\dim_{\mathrm{H}} \Lambda_{M,p}^{d} = \dim_{\mathrm{B}} \Lambda_{M,p}^{d} = \frac{\log(M^d \cdot p)}{\log M} \text{ a.s.}$$

The meaning of the nominator of the fraction in (1):

$$M^d \cdot p = \mathbb{E}\left[\#\mathcal{E}_1\right].$$

Therefore

(2) $$M^d \cdot p \leqslant 1 \Longrightarrow \Lambda = \emptyset \text{ a.s.}$$

We will prove this formula in the second later in the sace when $d = 1$.

(3)

$$\dim_{\mathrm{H}} \Lambda_{M,p}^{d} > 1 \text{ a.s. conditioned on non-extinction } \iff p > \frac{1}{M^{d-1}}.$$

Observe that $\#\mathcal{E}_n$ is a Galton-Watson Branching process with offspring distribution $\mathrm{Bin}(M^d, p)$. So, now we recall from Probability Theory:

1. If $M^d \cdot p = \mathbb{E}\left[\#\mathcal{E}_1\right] \leqslant 1$ then the process dies out in finite steps with probability 1.
2. If $M^d \cdot p = \mathbb{E}\left[\#\mathcal{E}_1\right] > 1$. Then the probability that the process does not die out is positive.

Now we give a formal defintion of the Fractal percolation. We confine ourselves to $d = 1$ that is we give the defintion on the line.

Given are $M \geqslant 2$ and $p \in (0, 1)$.

Let $\mathcal{T}$ be the M-adic tree. That is for each n, $\mathcal{T}$ has M^n nodes at level n, which we denote by strings $\underline{i}_n = i_1 \dots i_n$, where $i_k \in \{0, \dots, M-1\}$ for $k = 1, \dots, n$. There is one node at level 0 , the root, denoted $\emptyset$.

The set of "outcomes" Ω is the space of labeled trees, i.e., each node $i_1 \dots i_n$ obtains a label $X_{i_1 \dots i_n}$, which will be 0 or 1. Let $\mathcal{F}$ be the product σ-algebra on Ω. We consider a probability measure $\mathbb{P}_p$ on $(\Omega, \mathcal{F})$ by requiring that the $X_{i_1 \dots i_n}$ are independent Bernoulli random variables, with $\mathbb{P}_p \left(X_{\emptyset} = 1 \right) = 1$, and for $n \geqslant 1$ and $i_1 \dots i_n \in \{0, \dots, M-1\}^n$

$$\mathbb{P}_p \left(X_{i_1 \dots i_n} = 1 \right) = p. \tag{4}$$

The randomly labeled tree generates a random Cantor set in $[0,1]$ in the following way. Define

$$I_{i_1\dots i_n} := \left[\frac{i_1}{M} + \frac{i_2}{M^2} + \cdots + \frac{i_n}{M^n}, \frac{i_1}{M} + \frac{i_2}{M^2} + \cdots + \frac{i_n}{M^n} + \frac{i_n+1}{M^n}\right]. \tag{5}$$

The n-th level approximation Λ^n of the random Cantor set is a union of such n-th level M-adic intervals selected by the sets S_n defined by

$$S_n = \{i_1 \dots i_n : X_{i_1} = X_{i_1 i_2} = \cdots = X_{i_1\dots i_n} = 1\}. \tag{6}$$

Then

$$\mathcal{E}_n := \{I_{i_1\dots i_n} : (i_1 \dots i_n) \in S_n\}.$$

The random Cantor set $\Lambda = \Lambda^1_{M,p}$ is

(7) $$\Lambda = \bigcap_{n=1}^{\infty} \Lambda^n = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \dots i_n \in S_n} I_{i_1 \dots i_n}.$$

Let $Z_n = \mathrm{Card}\,(S_n)$ be the number of non-empty intervals $I_{i_1 \dots i_n}$ in Λ^n and let $Z_0 := 1$. Then $(Z_n)_{n \in \mathbb{N}}$ is a branching process with offspring distribution the law of Z_1, which is Binomial(M, p) Namely, let $\xi_i^{(n)}$, for $i, n \geqslant 1$ be i.i.d. random variables such that $\xi_i^{(n)} \stackrel{\mathrm{d}}{=} Z_1$. Then

$$Z_{n+1} := \begin{cases} \xi_1^{(n+1)} + \cdots + \xi_{Z_n}^{(n+1)}, & \text{if } Z_n > 0 \\ 0, & \text{if } Z_n = 0 \end{cases}$$

A simple consequence of the definition

Corollary 1.1

Given $M \geqslant 2$, $d \geqslant 1$ *and given the probabilities* $p, \widetilde{p}, \widehat{p} \in (0,1)$ *such that*

(8) $$\widehat{p} = p \cdot \widetilde{p}.$$

Now we consider the corresponding three Mandelbrot percolation sets:

$$\Lambda = \Lambda^d_{M,p}, \quad \widetilde{\Lambda} = \Lambda^d_{M,\widetilde{p}}, \quad \widehat{\Lambda} = \Lambda^d_{M,\widehat{p}}.$$

Then we have

(9) $$\widehat{\Lambda} \stackrel{d}{=} \Lambda \cap \widetilde{\Lambda}.$$

Proof in the case when $d = 1$.

For the probabilities $\widehat{p}, p$ and $\widetilde{p}$ and for every $n \geqslant 0$, $i_1 \dots i_n \in \{0, \dots, M-1\}^n$ we define the Bernoulli random variables $\widehat{X}_{i_1 \dots i_n}$, $X_{i_1 \dots i_n}$ and $\widetilde{X}_{i_1 \dots i_n}$ as in (4). Then by the assumption $\widehat{p} = p\widetilde{p}$

$$\widehat{X}_{i_1 \dots i_n} \stackrel{d}{=} X_{i_1 \dots i_n} \cdot \widetilde{X}_{i_1 \dots i_n}. \tag{10}$$

This implies that for every n

$$\widehat{S}_n \stackrel{d}{=} S_n \cap \widetilde{S}_n, \tag{11}$$

where the random sets $\widehat{S}_n, S_n$ and $\widetilde{S}_n$ are defined as in (6) for the probabilities $\widehat{p}, p$ and $\widetilde{p}$ respectively. This and (7) imply that (9) holds.

Theorem 2.1

The dimension formula for the homogeneous fractal percolation set $\Lambda^d_{M,p}$ is

(12)

$$\dim_{\mathrm{H}} \Lambda^d_{M,p} = \dim_{\mathrm{B}} \Lambda^d_{M,p} = \frac{\log(M^d \cdot p)}{\log M}$$ *a.s. conditioned on non-extinction.*

Recall: $M^d \cdot p = \mathbb{E}\left[\#\mathcal{E}_1\right]$. Now we prove (12).
Throughout the proof we always assume that $d = 1$. We fix a $p \in (0,1)$ and $M \geqslant 2$ The proof follows a lecture notes of Michel Dekking. Let

$$\Lambda := \Lambda^1_{M,p}.$$

$$q := \mathbb{P}\left(\Lambda = \emptyset\right), \quad I_k := \left[\frac{k}{M}, \frac{k+1}{M}\right], k = 0, \dots, M-1.$$

That is we prove in this one-dimensional setting, that the dimension of the frcatal percolation set Λ is $\frac{\log Mp}{M}$. We always assume that

$$p > \frac{1}{M} \tag{13}$$

otherwise $\Lambda = \emptyset$ a.s.. The following lemma will be important in the proof of the dimension formula.

Lemma 2.2

For every $\alpha > 0$ either $\mathcal{H}^{\alpha}(\Lambda) = 0$ holds a.s. or $\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right) = q$. In formula:

$$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right) \in \{q, 1\}. \tag{14}$$

Recall: $q = \mathbb{P}(\Lambda = \emptyset)$.

Proof

Let

$$Z_n := \#S_n.$$

Consider the probability generator function of Z_1

$$g(s) := \mathbb{E}\left[s^{Z_1}\right] = \sum_{k=0}^{\infty} p_k s^k, \qquad p_k = \mathbb{P}\left(Z_1 = k\right).$$

On the next slide we prove that $\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right)$ is a fixed point of g.

Using that the set of fixed points of g consists of 1 and q, this will complete the proof. So the calculation is as follows:

Proof cont.

$$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda)=0\right)=\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda_0)=0,\dots,\mathcal{H}^{\alpha}(\Lambda_{M-1})=0\right)$$

$$=\sum_{k=0}^{M}\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda_i)=0,\forall i=0,\dots,M-1|Z_1=k\right)\cdot\mathbb{P}\left(Z_1=k\right)$$

$$=\sum_{k=0}^{M}\left[\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda_0)=0|Z_1=k\right)\right]^k\cdot\mathbb{P}\left(Z_1=k\right)$$

$$=\sum_{k=0}^{M}\left[\mathbb{P}\left(\left(\frac{1}{M}\right)^{\alpha}\mathcal{H}^{\alpha}(\Lambda)=0\right)\right]^k\cdot\mathbb{P}\left(Z_1=k\right)$$

$$=\sum_{k=0}^{M}\left[\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda)=0\right)\right]^k\cdot\mathbb{P}\left(Z_1=k\right)=g\left(\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda)=0\right)\right).\square$$

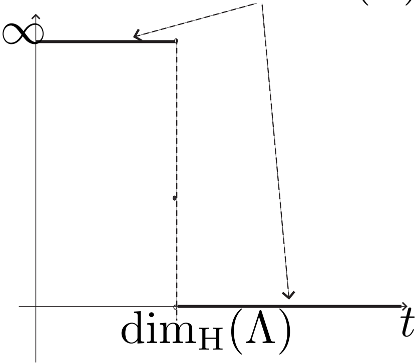
Lemma 2.3

The random variable $\dim_{\mathrm{H}} \Lambda$ *is almost surely* *constant* *on the event* $\{\Lambda \neq \emptyset\}$.

Namely, if $\mathbb{E}\left[Z_1\right] \leqslant 1$ then $\Lambda = \emptyset$ almost surely. We assume that $\mathbb{E}\left[Z_1\right] > 1$. Let $(\Omega, \mathcal{F}, \mathbb{P}_p)$ be the ambient probability space on which the random set Λ is defined. Let $\Omega^{\emptyset}$ be defined s.t. if $\omega \in \Omega^{\emptyset}$ then $\Lambda = \emptyset$. Clearly $\mathbb{P}\left(\Omega^{\emptyset}\right) = q$. If $\dim_{\mathrm{H}} \Lambda$ is not a constant almost surely conditioned on non-extinction then then $\exists \Omega_1, \Omega_2 \subset \Omega \backslash \Omega^{\emptyset}$ with $0 < \mathbb{P}\left(\Omega_i\right) < 1 - q$ and $\gamma_1 < \gamma_2$ such that for $\omega \in \Omega_1$ $\dim_{\mathrm{H}} \Lambda(\omega) < \gamma_1$ and for $\omega \in \Omega_2$, $\dim_{\mathrm{H}} \Lambda(\omega) > \gamma_2$. For $\omega \in \Omega_1 \cup \Omega^{\emptyset}$, $\mathcal{H}^{\gamma_1}(\Lambda) = 0$ and for $\omega \in \Omega_2$, $\mathcal{H}^{\gamma_1}(\Lambda) = \infty$. Then $\mathbb{P}\left(\mathcal{H}^{\gamma_1}(\Lambda) = 0\right) \in (q, 1)$ which is impossible by Lemma 2.2.

$$(15)\qquad \mathcal{H}^t(\Lambda) = \lim_{\delta\to 0}\left\{\underbrace{\inf\left\{\sum_{i=1}^{\infty}|A_i|^t : \Lambda\subset\bigcup_{i=1}^{\infty}A_i;\ |A_i|<\delta\right\}}_{\mathcal{H}^s_\delta(A)}\right\}.$$

$t\to\mathcal{H}^t(\Lambda)$



Above: $\Lambda\subset\mathbb{R}^d$, $t\geqslant 0$,

The Hausdorff dimension of Λ

$$\begin{aligned}\dim_{\mathrm{H}}(\Lambda) &= \inf\left\{t:\mathcal{H}^t(\Lambda)=0\right\}\\ &= \sup\left\{t:\mathcal{H}^t(\Lambda)=\infty\right\}.\end{aligned}$$

The upper bound

Λ_n consists of Z_n intervals of length M^{-n}. This implies that

$$\mathcal{H}^{\alpha}_{M^{-n}}(\Lambda) \leqslant Z_n \cdot \left(M^{-n}\right)^{\alpha}. \tag{16}$$

By Markov inequality:

$$\mathbb{P}\left(\mathcal{H}^{\alpha}_{M^{-n}}(\Lambda) \geqslant \varepsilon\right) \leqslant \frac{\mathbb{E}\left[\mathcal{H}^{\alpha}_{M^{-n}}(\Lambda)\right]}{\varepsilon} \overset{(16)}{\leqslant} \frac{\mathbb{E}\left[Z_n\right]}{\varepsilon M^{n\alpha}} = \frac{\mathbb{E}\left[Z_1\right]^n}{\varepsilon M^{n\alpha}} = \frac{1}{\varepsilon}\left(\frac{\mathbb{E}\left[Z_1\right]}{M^{\alpha}}\right)^n \tag{17}$$

The upper bound cont.

Let

$$\alpha > \frac{\log \mathbb{E}\left[Z_1\right]}{\log M} = \frac{\log(Mp)}{\log M}.$$

Then

$$\mathbb{E}\left[Z_1\right] < M^{\alpha}.$$

Using Borel Cantelli and (17) this means that

$$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right) = 1,$$

since $\lim_{n\to\infty} \mathcal{H}^{\alpha}_{M^{-n}} = \mathcal{H}^{\alpha}(\Lambda)$. That is $\dim_{\mathrm{H}} \Lambda \leqslant \alpha$ a.s.$\square$

The lower bound

Let

$$s := \frac{\log(Mp)}{\log M}.$$

We want to prove that

(18) $\dim_{\mathrm{H}} \Lambda \geqslant s$ a.s. conditioned on nonextinction.

First we prove that

Lemma 2.4

If $B \subset [0,1]$ has the property that $\mathbb{P}(\Lambda \cap B \neq 0) > 0$ then this implies that $\dim_{\mathrm{H}} B \geqslant \frac{-\log p}{M}$.

The lower bound cont.

Proof of the Lemma 2.4 slide I

Recall that in the definition of the Hausdorff dimension we can restrict ourselves to covers by M-adic intervals like $I := \left[\frac{k-1}{M^n}, \frac{k}{M^n}\right]$. If I is such an interval and I_{left}, I_{right} are its neighbours then

$$\mathbb{P}(\Lambda \cap I \neq \emptyset) \leqslant \sum_{J \in \{I_{\text{left}}, I, I_{\text{right}}\}} \mathbb{P}(J \text{ is selected }) = 3p^n$$

Using that $|I| = M^{-n}$ and the solution of the equation $p^n = (M^{-n})^x$ is $x = \frac{-\log p}{\log M}$, from the previous formula we get that

(19) $$\mathbb{P}(\Lambda \cap I \neq \emptyset) \leqslant 3|I|^{\frac{-\log p}{\log M}}.$$

The lower bound cont.

Proof of the Lemma 2.4 slide II

To prove that $\dim_{\mathrm{H}} B \geqslant \frac{-\log p}{\log M}$ it is enough to verify that there exists a constant $C > 0$ such that for an arbitrary covering $\{I_k\}$ of Λ by M-adic intervals (not necessarily of the same length) we have

(20) $$\sum_k |I_k|^{\frac{-\log p}{\log M}} > C > 0.$$

To see this we define $C := \mathbb{P}(\Lambda \cap B)$. By assumption $C > 0$. Using that I_k is a cover of B we have:

The lower bound cont.

Proof of the Lemma 2.4 slide III

$$0 < C = \mathbb{P}\left(\Lambda \cap B \neq \emptyset\right) \leqslant \mathbb{P}\left(\Lambda \cap \bigcup_k I_k \neq \emptyset\right)$$

$$\leqslant \sum_k \mathbb{P}\left(\Lambda \cap I_k \neq \emptyset\right) \overset{(19)}{\leqslant} \sum_k 3|I_k|^{\frac{-\log p}{\log M}}.$$

This completes the proof of the Lemma.

The lower bound cont.

Given $p, \widetilde{p}, \widehat{p} \in (0,1)$ such that

(21) $$\widehat{p} = p \cdot \widetilde{p}.$$

Now we consider the corresponding three Mandelbrot percolation sets:

$$\Lambda = \Lambda^1_{M,p}, \quad \widetilde{\Lambda} = \Lambda^1_{M,\widetilde{p}}, \quad \widehat{\Lambda} = \Lambda^1_{M,\widehat{p}}.$$

We have proved in Corollary 1.1 that

(22) $$\widehat{\Lambda} \stackrel{d}{=} \Lambda \cap \widetilde{\Lambda}.$$

In particular,

The lower bound cont.

$$\mathbb{P}_{\widehat{p}}\left(\widehat{\Lambda} \neq \emptyset\right) = (\mathbb{P}_p \times \mathbb{P}_{\widetilde{p}})\left(\Lambda \cap \widetilde{\Lambda} \neq \emptyset\right). \tag{23}$$

Let

$$V_{p,\widetilde{p}} := \left\{\omega_p \in \Omega_p : \mathbb{P}_{\widetilde{p}}\left(\omega_{\widetilde{p}} \in \Omega_{\widetilde{p}} : \Lambda(\omega_p) \cap \widetilde{\Lambda}(\omega_{\widetilde{p}}) \neq \emptyset\right) > 0\right\}.$$

The lower bound cont.

Lemma 2.5

Assume that $\widehat{p} > \frac{1}{M}$. *We choose* $p, \widetilde{p}$ *such that (as always)* $\widehat{p} = p \cdot \widetilde{p}$. *Then*

(24) $$\mathbb{P}_p \left(V_{p,\widetilde{p}} \right) > 0.$$

Proof.

By assumption $\mathbb{P}_{\widehat{p}} \left(\widehat{\Lambda} \neq \emptyset \right) > 0$. Then the assertion of the Lemma follows from (23) and Fubini Theorem. □

The lower bound cont.

Here we use the notation and assumption of Lemma 2.5. Now we fix an $\omega_p \in V_{p,\widetilde{p}}$. Let $B := \Lambda(\omega_p)$. Then by the definition of $V_{p,\widetilde{p}}$ we have

$$\mathbb{P}_{\widetilde{p}}\left(\omega_{\widetilde{p}} \in \Omega_{\widetilde{p}} : \Lambda(\omega_{\widetilde{p}}) \cap B \neq \emptyset\right) > 0.$$

This implies by Lemma 2.4 that

(25) $$\dim_{\mathrm{H}} \Lambda(\omega_p) \geqslant \frac{-\log \widetilde{p}}{\log M} \quad \text{for all } \omega_p \in V_{p,\widetilde{p}}.$$

The lower bound cont.

We have assumed that

$$\frac{1}{M} < \widehat{p} = p \cdot \widetilde{p}.$$

That is $\widetilde{p} > \frac{1}{Mp}$ and $\widetilde{p}$ can be as close to $\frac{1}{Mp}$ as we want. So on a set of positive $\mathbb{P}_p$-measure of $\omega_p \in V_{p,\widetilde{p}}$, we have

$$\frac{-\log \widetilde{p}}{\log M} \overset{(25)}{\leqslant} \dim_{\mathrm{H}} \Lambda(\omega_p) \leqslant \frac{\log(Mp)}{\log M}. \tag{26}$$

But by Lemma 2.3 we know that $\dim_{\mathrm{H}} \Lambda_p(\omega_p)$ is almost surely constant on $\Lambda_p \neq \emptyset$. This completes the proof.□