Home work assignments for the Stochastic Processes course, 2023

Many of the exercises are assigned from my favorite first reading book on Stochastic processes. Entitled: Essentials of Stochastic Processes (Almost final version of the 2nd edition, December 2011) by Rick Durrett. This book is freely available on the authors home page: Click here or type in the browser:

`https://services.math.duke.edu/~rtd/EOSP/EOSP2E.pdf`

Please, never send homework assignments by email.

First homework assignment. **Due at 10:15 on 15 September 2023**
R. Durrett, Essentials of Stochastic Processes: Chapter 1.12, Exercises:
1.1, 1.2, 1.3, 1.6, 1.7, 1.8 (c),(d), 1.10 (a), 1.11, 1.12 (a),(b), 1.13, 1.14, 1.15, 1.16.

Second homework assignment. **Due at 10:15 on 29 September 2023.**
R. Durrett, Essentials of Stochastic Processes: Chapter 1.12, Exercises:
1.21 , 1.26, 1.31, 1.36, 1.37, 1.41, 1.46, 1.47, 156, 1.58, 1.59,1.62, 1.63

Third homework assignment. **Due at 10:15 on 6 October 2023.**
R. Durrett, Essentials of Stochastic Processes: Chapter 1.12, Exercises:
1.67, 1.68, 1.69, 1.72, 1.73, 1.74, 1.76

Fourth homework assignment. **Due at 10:15 on 13 October 2023.**

1. Consider the unit interval $I := [0,1]$. Moreover, for every n and $(i_1,\dots,i_n) \in \{0,1,2\}^n$ we consider the interval $I_{i_1\dots i_n} \subset I$ which is the set of those numbers whose base 3 expansion starts with $(i_1 \dots i_n)$. That is
$$I_{i_1\dots i_n} := \left[\sum_{k=1}^{n} \frac{i_k}{3^n}, \frac{1}{3^n} + \sum_{k=1}^{n} \frac{i_k}{3^n} \right].$$
Let X_0, X_1, X_2 be independent Bernoulli(p_0), Bernoulli(p_1) and Bernoulli(p_2) random variables respectively. That is $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$ for $i = 0,1,2$. Moreover for every n and $(i_1,\dots,i_n) \in \{0,1,2\}^n$ we are given the random variables $X_{i_1\dots i_n}$ such that on the one hand $\{X_{i_1\dots i_n}\}_{n\geq 1,(i_1,\dots,i_n)\in\{0,1,2\}^n}$ are independent and on the other hand: $X_{i_1\dots i_n} \stackrel{d}{=} X_{i_n}$. For every $n \geq 1$ we define the set $E_n \subset [0,1]$ by
$$E_n := \bigcup_{X_{i_1}\cdot X_{i_1,i_2}\cdots X_{i_1,i_2,\dots,i_n}=1} I_{i_1\dots i_n}.$$
Finally, we define the set $E := \bigcap_{n=1}^{\infty} E_n$. Assume that $p_0 = \frac{2}{3}, p_1 = \frac{3}{4}$ and $p_2 = \frac{1}{2}$. Question: Is it true that $\mathbb{P}(E \neq \emptyset) > 0$

2. Given a branching process with the following offspring distribution, determine the extinction probabilities q:

   (a) $p_0 = 0.25, p_1 = 0.4, p_2 = 0.35,\ p_n = 0$ if $n \geq 3$,

   (b) $p_0 = 0.5, p_1 = 0.1, p_2 = 0, p_3 = 0.4,\ p_n = 0$ if $n \geq 4$.

3. Consider the branching process with offspring distribution as in the previous exercise part (b). What is the probability that the population is extinct in the second generation $X_2 = 0$, given that it did not die out in the first generation?

4. Consider the branching process with offspring distribution given by $\{p_n\}_{n=0}^{\infty}$. We change this process into an irreducible Markov chain by the following modification:

   whenever the population dies out, then the next generation has exactly one new individual. That is $\mathbb{P}(X_{n+1} = 1 | X_n = 0) = p(0,1) = 1$. For which $\{p_n\}_{n=0}^{\infty}$ will this chain be null recurrent, recurrent, transient?

5. Let $X_1, X_2, \dots$ i.i.d. random variables taking values in the integers such that $\mathbb{E}[X_i] = 0$ for all i. Let $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$.

   (a) Let $G_n(x) := \sum_{j=0}^{n} \mathbb{1}_{\{S_j = x\}}$. That is $G_n(x)$ is the expected number of visits to x in the first n steps. Show that for all n and x, $G_n(0) \geq G_n(x)$. (Hint: consider the first j with $S_j = x$.)

   (b) Note that the Law of Large Numbers implies that for each $\varepsilon > 0$ we have: $\lim_{n\to\infty} \mathbb{P}(|S_n| \leq n\varepsilon) = 1$. Using this prove that for each $\varepsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n} \sum_{|x| \leq \varepsilon \cdot n, x \in \mathbb{Z}} G_n(x) = 1$

   (c) Using (a) and (b), show that for each $M < \infty$ there is an n such that $G_n(0) \geq M$.

   (d) Now prove that S_n is a recurrent Markov chain.

Fifth homework assignment. **Due at 12:00 on 20 October 2023.**
R. Durrett, Essentials of Stochastic Processes: Chapter 2.6, Exercises:
2.1 (a),(b), 2.6, 2.9 (a), 2.16 (a),(b), 2.17 (a) (b) (c), 2.21, 2.22, 2.25, 2.27, 2.28

Sixth homework assignment. **Due at 10:15 on 27 October 2023.**
R. Durrett, Essentials of Stochastic Processes: Chapter 2.6, Exercises:
2.29, 2.31, 2.32, 2.33, 2.44, 2.45 2.50

Seventh homework assignment. **Due at 10:15 on 10 November 2023.**
R. Durrett, Essentials of Stochastic Processes: Chapter 4.8, Exercises: 4.2, 4.3, 4.4, 4.8, 4.10, 4.11, 4.14, 4.16, 4.17, 4.19, 4.20

Eight homework assignment. **Due at 10:15 on 17 November 2023.**
R. Durrett, Essentials of Stochastic Processes:

1. If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X given that $X + Y = m$.

2. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

3. Consider n independent trials, each of which results in one of the outcomes $\{1, ..., k\}$, with respective probabilities $\{p_1, ..., p_k\}$, $\sum_{i=1}^{k} p_i = 1$. Let N_i denote the number of trials that result in outcome i, $i = 1, ..., k$. For $i \neq j$ find $\mathbb{E}[N_i | N_j > 0]$.

4. Let U be a uniform random variable on $(0, 1)$, and suppose that the conditional distribution of X, given that $U = p$, is binomial with parameters n and p. Find the probability mass function of X. That is find for all $0 \leq i \leq n$, $\mathbb{P}(X = i) = ?$

Hint: In the solution of this problem you may want to use the following general formula: Let E be an event and Y be a continuous r.v. with density function: $f_Y(y)$. Then:

$$\mathbb{P}(E) = \int_{-\infty}^{\infty} P(E|Y=y) f_Y(y) dy.$$

Moreover you may also want to use the following formula:

$$\int_0^1 p^i (1-p)^{n-i} dp = \frac{i!(n-i)!}{(n+1)!}$$

5. The joint density of X and Y is given by $f(x,y) = \frac{e^{-x/y}e^{-y}}{y}$, $0 < x < \infty$, $0 < y < \infty$. Compute $\mathbb{E}[X^2|Y] =?$

Ninth homework assignment. **Due at 10:15 on December 1 2023**

R. Durrett, Essentials of Stochastic Processes: Chapter 5.6 Exercises: 5.2, 5.3, 5.6, 5.7, 5.8, 5.9, 5.10

Tenth homework assignment. **Due at 10:15 on 8 December 2023.**

In some of the exercises you may want to use the following fact: If random variables X and Y are jointly normal and $\mathrm{Cov}(X,Y) = 0$ then X and Y are independent.

1. Let $Z \sim \mathcal{N}(0,1)$. We define X_t for all $t \geq 0$ by $X_t = \sqrt{t} \cdot Z$. Then the stochastic process $X = \{X_t : t \geq 0\}$ has continuous path and for all $t \geq 0$ we have $X_i \sim \mathcal{N}(0,t)$. Is X_t a Brownian motion? (Check if all the conditions (a)-(c) on slide 9 from File F hold for X_t. In particular, the variance of the increments.)
2. Let $B(t)$ be the one-dimensional Brownian motion. Show that $\mathrm{Cov}(B(t), B(s)) = \min\{s,t\}$.
3. Let $B(t)$ be the one-dimensional Brownian motion. Fix an arbitrary positive number s. Show that the process $B(t+s) - B(s)$ is also Brownian motion.
4. Let $B(t)$ be the one-dimensional Brownian motion. Show that the process $-B(t)$ is also Brownian motion.
5. Let $B(t)$ be the one-dimensional Brownian motion. Fix a positive number a. Prove that $a^{-1/2}B(at)$ is is also Brownian motion.
6. Let $B(t)$ be the one-dimensional Brownian motion. Consider the following stochastic process: $V(0) :=$ and $V(t) = tB(1/t)$. Prove that $V(t)$ is also a Brownian motion.
7. Let $B(t)$ and $\widetilde{B}(t)$ be two independent Brownian motions and let $\rho \in (0,1)$ We define $X(t) := \rho B(t) + \sqrt{1-\rho^2}\widetilde{B}(t)$. Prove that $X(t)$ is also a Brownian motion.