Recall: $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, $\alpha > 0$.

$\Gamma(n) = (n-1)!$ in particular $\Gamma(1) = 1$.

Gamma(α,λ) distribution density function:

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Beta(a,b) distribution

Let $a, b > 0$. The Beta(a,b) distribution is a continuous distribution on the interval $[0,1]$. First we define

$$B(a,b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

For example:

$$B(1,n) = \frac{\Gamma(1+n)}{\Gamma(1)\Gamma(n)} = \frac{n!}{1\cdot(n-1)!} = n.$$

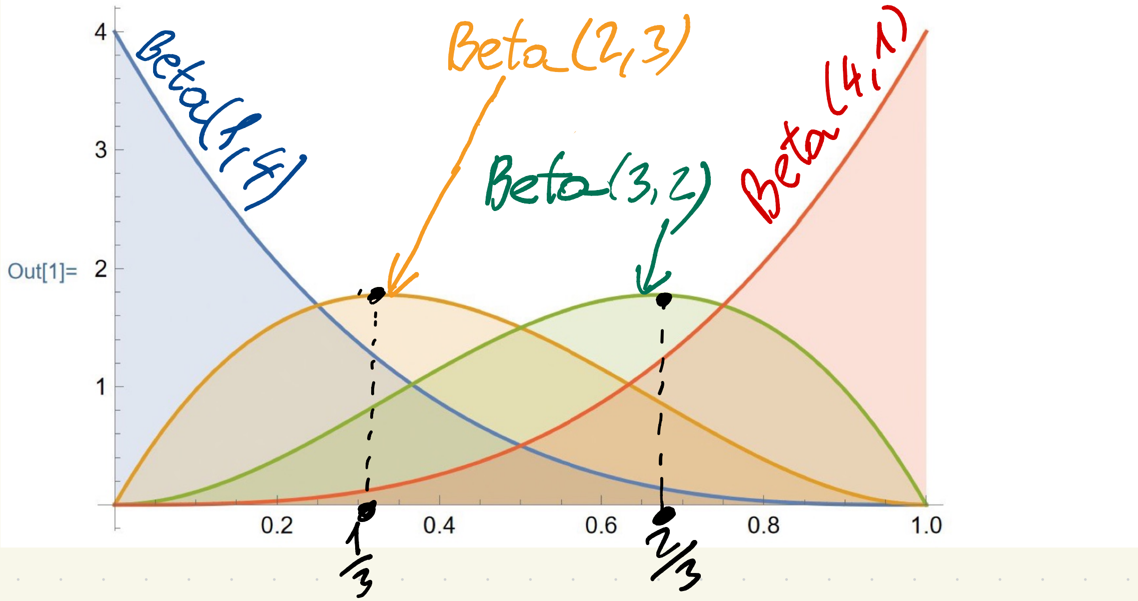
The density function of the Beta(a,b) distribution:

$$f_{a,b}(x) = \frac{1}{B(a,b)} \cdot x^{a-1} \cdot (1-x)^{b-1} \quad \boxed{x \in [0,1]}$$

If $x \notin [0,1]$ then $f_{a,b}(x) := 0$.

```
In[1]:= Plot[Table[PDF[BetaDistribution[a, 5 - a], x], {a, {1, 2, 3, 4}}]
          // Evaluate, {x, 0, 1}, Filling → Axis]
```

Out[1]=



$X \sim \text{Beta}(a,b)$; $E[X] = \dfrac{\alpha}{\alpha+\beta}$; $Var(X) = \dfrac{ab}{(a+b)^2(1+a+b)}$.

The mode of Beta(a,b) Let $a>1$, $b>1$.

The mode is the most likely value (corresponding to the peak of the distribution function). This is $\dfrac{\alpha-1}{\alpha+\beta-2}$.

The median of Beta(a,b)

$$F_{a,b}(y) = \int_0^y x^{a-1}(1-x)^{b-1}\,dx.$$

Median $= F_{a,b}^{-1}\left(\frac{1}{2}\right)$ no closed formula

For $a=1$, $b>0$, Median $= 1-2^{-1/\beta}$.

Let

$X_1, \dots, X_n$ i.i.d. $X_i \sim$ Uniform$(0,1)$.

Let $X_{(j)}$ be the j-th smallest of $X_1, \dots, X_n$. Then $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$.

Then

$$X_{(j)} \sim \text{Beta}(j, n+1-j).$$

5. Let $X_1, X_2, \ldots$ i.i.d. random variables taking values in the integers such that $\mathbb{E}[X_i] = 0$ for all i. Let $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$.

$\mathbb{E}[\sum_{j=0}^{n} \mathbb{1}_{\{S_j=x\}}]$

(a) Let $G_n(x) := \cancel{\sum_{j=0}^{n} \mathbb{1}_{\{S_j=x\}}}$. That is $G_n(x)$ is the expected number of visits to x in the first n steps. Show that for all n and x, $G_n(0) \geq G_n(x)$. (Hint: consider the first j with $S_j = x$.)

(b) Note that the Law of Large Numbers implies that for each $\varepsilon > 0$ we have: $\lim_{n\to\infty} \mathbb{P}(|S_n| \leq n\varepsilon) = 1$. Using this prove that for each $\varepsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n} \sum_{|x|\leq\varepsilon\cdot n, x\in\mathbb{Z}} G_n(x) = 1$

(c) Using (a) and (b), show that for each $M < \infty$ there is an n such that $G_n(0) \geq M$.

(d) Now prove that S_n is a recurrent Markov chain.

Solution (a) $G_n(x) = \mathbb{E}[\sum_{j=0}^{n} \mathbb{1}_{\{S_j=x\}}]$. $G_n(0) \geq G_n(x)$.

Clearly, for $\forall x \in \mathbb{Z}$: $(*)$ $G_k(x) \leq G_\ell(x)$ if $k \leq \ell$

Let $S \subseteq \mathbb{Z}$ be the state space. If $x \in \mathbb{Z} \setminus S$ then $G_n(x) = 0$ $\forall n \in \{0,1,2,\ldots\}$. So, we may assume that $x \in S$. Let $T_x := \min\{k \geq 1 : S_k = x\}$. $G_n(x) =$

$$= \mathbb{E}\left[\sum_{j=0}^{n} \mathbb{1}_{\{S_j=x\}}\right] = \sum_{j=1}^{n} \underbrace{\mathbb{E}\left[\sum_{k=j}^{n} \mathbb{1}_{\{S_k=x\}} \,\Big|\, T_x = j\right]}_{G_{n-j}(0)} \cdot \mathbb{P}(T_x = j)$$

$$= \sum_{j=1}^{\infty} \overset{\leq G_n(0)}{G_{n-j}(0)} \cdot \mathbb{P}(T_x = j) \overset{(*)}{\leq} G_n(0).$$ This is so since

$\sum_{j=1}^{\infty} \mathbb{P}(T_x = j) \leq 1$ & $G_{n-j}(0) \leq G_n(0)$. This completes the proof of (a).

(b) By L.L.N. (Law of Large Numbers) for every $\varepsilon > 0$ we have

$(**)$ $\lim_{n\to\infty} \mathbb{P}(|S_n| \leq n\varepsilon) = 1$. Namely, $\mathbb{E}[X_n] = 0$ $\forall n$.

Hence by the L.L.N. we have

$$\lim_{n\to\infty} \frac{\overbrace{X_1+\dots+X_n}^{S_n}}{n} = \mathbb{E}[X_1] = 0 \text{ a.s.}$$ Hence, for $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) = 0.$$ This verifies that (**) holds.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{|x|\le \varepsilon n} G_n(x) = 1. \quad (***)$$

Now we prove:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{|x|\le\varepsilon n} \underbrace{\mathbb{E}\left[\sum_{j=0}^{n} \mathbb{1}_{\{S_j=x\}}\right]}_{G_n(x)} =$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \underbrace{\sum_{|x|\le\varepsilon n} \underbrace{\mathbb{E}[\mathbb{1}_{\{S_j=x\}}]}_{\mathbb{P}(S_j=x)}}_{\mathbb{P}(|S_j|\le\varepsilon n)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \mathbb{P}(|S_j|\le\varepsilon n).$$ Now we prove that

$$\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \mathbb{P}(|S_j|\le\varepsilon n) = 1.} \quad (****)$$ Let $a_{j,n} := \mathbb{P}(|S_j|\le\varepsilon n)$.

We know that $\lim_{n\to\infty} a_{n,n} = 1$ & $a_{j,n} \le 1$.
(from (**))

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} a_{j,j} = 1$$

$$a_{j,j} = \mathbb{P}(|S_j| \le \varepsilon j) \le \mathbb{P}(|S_j|\le\varepsilon n) = a_{j,n}$$ if $j \le n$

$$1 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} a_{j,j} \le \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} a_{j,n} \le 1.$$

Hence $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{j=0}^{n} a_{j,n} = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{j=0}^{n} \mathbb{P}(|S_j| \le \varepsilon n) = 1.$

This proves that (****) holds. Hence (***) also holds. This completes the proof of part (b).

(c) We want to prove that $\forall M>0, \exists n$ s.t $G_n(0) \ge M$.

Recall that we have proved that $\forall \varepsilon > 0$:
$$\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{|x|\le \varepsilon n} G_n(x) = 1.} \quad (***)$$

$$1 = \lim_{n\to\infty} \frac{1}{n} \sum_{|x|\le \varepsilon n} G_n(x) \le \underline{\lim}_{n\to\infty} \frac{1}{n} \cdot 2\varepsilon n \, G_n(0) = 2\varepsilon \, \underline{\lim}_{n\to\infty} G_n(0)$$

That is $\forall \varepsilon$, $\exists n_0$ s.t. for all $n \ge n_0$ $G_{n_0}(0) \ge \frac{1}{2\varepsilon}$.

We choose $\varepsilon > 0$ s.t. $\frac{1}{2\varepsilon} > M$. Then we have proved the assertion of part (c).

(d) It is obvious that S_n is irreducible. Hence it is enough to prove that 0 is recurrent. We have seen in part (c) that the expected number of visits to 0 in the first n steps tends to infinity as $n \to \infty$.

Reflecting Random Walk

This is a Discrete time MC.

$\{X_n\}_{n=0}^{\infty}$ takes values from $\mathbb{N} = \{0, 1, 2, \dots\}$.

$$\left.\begin{array}{l} p(i, i+1) = p \quad \text{for } i \geq 0 \\ p(i, i-1) = 1-p \quad \text{for } i \geq 1 \\ p(0,0) = 1-p \end{array}\right\}$$

This is a birth & death chain. So, the stationary distribution (if exists) can be found by the detailed balance equation:

$$p\,\pi(i) = (1-p)\,\pi(i+1) \quad \text{for } i \geq 0.$$

This yields $\pi(i) = c \cdot \left(\frac{p}{1+p}\right)$ for $c = \pi(0)$.

(a) $\boxed{p < \frac{1}{2}}$ Then $\frac{p}{1+p} < 1$, $\sum_{k=0}^{\infty} \left(\frac{p}{1+p}\right)^k = \frac{1}{1 - \frac{p}{1+p}}$

$$\pi(i) = \frac{1-2p}{1-p} \left(\frac{p}{1-p}\right)^i$$

the first return time to 0

Then $P(X_n = j) \to \pi(j)$; $E_0[T_0] = \frac{1}{\pi(0)} = \frac{1-p}{1-2p}$

This is so since the chain is aperiodic.

(b) $\boxed{p > \frac{1}{2}}$ The series above is divergent no stationary distribution

(c) $\boxed{p = \frac{1}{2}}$ The chain is null recurrent.
See Durrett's book p. 55.

$p < \frac{1}{2}$ positive recurrent

$p = 0$ null recurrent

$p > \frac{1}{2}$ transient.

Theorem (see Durrett's book p. 55 Theorem 1.23

For an irreducible chain the following are equivalent:

(i) There is a positive recurrent state

(ii) The stationary distribution exists

(iii) All states are positive recurrent.