2.28. The number of hours between successive trains is T which is uniformly distributed between 1 and 2. Passengers arrive at the station according to a Poisson process with rate 24 per hour. Let X denote the number of people who get on a train. Find (a) EX, (b) $\operatorname{var}(X)$.

(a) S is the time when the train arrives. $S \sim Uni(1,2)$.

$N(t)$: the number of passengers arrived at the station by time t.

The number of passengers on the train $X = N(S)$. By assumption $N(t) = Poi(24t)$.

S & N(t) independent

$$\mathbb{E}[X \mid S=t] = \mathbb{E}[N(t) \mid S=t] \stackrel{}{=} \mathbb{E}[N(t)] \stackrel{}{=} 24t.$$

$$\boxed{\mathbb{E}[X \mid S] = \mathbb{E}[N(S) \mid S] = 24S} \quad (*)$$

$$\mathbb{E}[X] = \int_{t=-\infty}^{\infty} \mathbb{E}[X \mid S=t] \cdot f_S(t)\,dt = \int_{t=1}^{2} \underbrace{\mathbb{E}[X \mid S=t]}_{24t} \cdot 1\,dt$$

$$= \left[12t^2\right]_1^2 = 12(4-1) = \underline{\underline{36}}$$

(b)

N(t) & S independent

$$\operatorname{Var}(X \mid S=t) = \operatorname{Var}(N(t) \mid S=t) = \operatorname{Var}(N(t)) = 24t$$

$N(t) \sim Poi(24t)$

$$(**) \quad \operatorname{Var}(X \mid S) = \operatorname{Var}(N(S) \mid S) = 24S$$

Conditional variance formula.

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid S)] + \mathrm{Var}(\mathbb{E}[X \mid S]).$$

$$\mathrm{Var}(X) = \mathbb{E}[24S] + \mathrm{Var}(24S)$$

(**) (*)

$\boxed{S \sim \mathrm{Uni}(1,2)}$

$$= 24\underbrace{\mathbb{E}[S]}_{3/2} + 24^2 \cdot \underbrace{\mathrm{Var}(S)}_{\frac{1}{12}\cdot(2-1)^2 = \frac{1}{12}} = 3 \cdot 12 + \frac{2^2 \cdot 12^{\cancel{2}}}{\cancel{12}} = 36 + 48$$

$$= \underline{\underline{84}}$$